Automorphic algebras of dynamical systems and generalised Inönü-Wigner contractions

Автоморфные алгебры динамических систем и обобщенные контракции Иненю-Вигнера

A. Karabanov

Cryogenic Ltd,
London, W3 7QE, UK

karabanov@hotmail.co.uk

А. Карабанов

ООО «Криогеника»,
г. Лондон, W3 7QE, Великобритания

karabanov@hotmail.co.uk

Abstract

Lie algebras $\mathfrak{a}$ with a complex underlying vector space V are studied that are automorphic with respect to a given linear dynamical system on V, i.e., a 1-parameter subgroup $G_t \subset Aut(\mathfrak{a}) \subset GL(V)$. Each automorphic algebra imparts a Lie algebraic structure to the vector space of trajectories of the group G_t. The basics of the general structure of automorphic algebras $\mathfrak{a}$ are described in terms of the eigenspace decomposition of the operator $M \in \mathfrak{Der}(\mathfrak{a})$ that determines the dynamics. Symmetries encoded by the presence of non-abelian automorphic algebras are pointed out connected to conservation laws, spectral relations and root systems. It is shown that, for a given dynamics G_t, automorphic algebras can be found via a limit transition in the space of Lie algebras on V along the trajectories of the group G_t itself. This procedure generalises the well-known Inönü-Wigner contraction and links adjoint representations of automorphic algebras to isospectral Lax representations on $\mathfrak{gl}(V)$. These results can be applied to physically important symmetry groups and their representations, including classical and relativistic mechanics, open quantum dynamics and nonlinear evolution equations. Simple examples are given.

Аннотация

Изучаются алгебры Ли $\mathfrak{a}$ с комплексным базовым векторным пространством V, автоморфные относительно заданной линейной динамической системы на V , т. е. 1-параметрической подгруппы $G_t \subset Aut(\mathfrak{a}) \subset GL(V)$. Каждая автоморфная алгебра сообщает Ли-алгебраическую структуру векторному пространству траекторий группы G_t. Основы общей структуры автоморфных алгебр $\mathfrak{a}$ описаны в терминах разложения по собственным подпространствам оператора $M \in \mathfrak{Der}(\mathfrak{a})$, определяющего динамику. Указаны симметрии, кодируемые наличием неабелевых автоморфных алгебр, связанные с законами сохранения, спектральными соотношениями и системами корней. Показано, что при заданной динамике G_t автоморфные алгебры могут быть найдены посредством предельного перехода в пространстве алгебр Ли на V вдоль траекторий самой группы G_t. Эта процедура обобщает известную контракцию Иненю-Вигнера и связывает присоединенные представления автоморфных алгебр с изоспектральными представлениями Лакса на $\mathfrak{gl}(V)$. Полученные результаты можно применить к физически важным группам симметрии и их представлениям, включая классическую и релятивистскую механику, открытую квантовую динамику и нелинейные эволюционные уравнения. Приведены простые примеры.

Keywords:

automorphic algebras, dynamical systems, generalised Inönü-Wigner contractions

Ключевые слова:

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Introduction

Lie groups and Lie algebras are a powerful mathematical tool that has a variety of physical applications. The local properties of a Lie group are described in terms of its Lie algebra. Lie algebras have also applications, fully separate from Lie groups. This makes the theory of Lie algebras independently useful. Finite-dimensional complex and real semisimple Lie algebras and their representations are fully classified [1–3].

The modern theory of Lie algebras mostly concerns infinite-dimensional generalisations (with links to modern problems of theoretical physics [4]) and geometric extensions (with links to algebraic groups and algebraic topology [5, 6]). Automorphisms of Lie algebras (and adjacent algebraic structures) describe the algebra symmetries and so play an important role in the theory. Normally, the direct problem is tackled, i.e., the problem of finding the group of automorphisms of a given Lie algebra. We address here the inverse problem — the problem of description of Lie algebras that have a given group of automorphisms.

The inverse problem is quite useful and intricate as well, even in the finite-dimensional case. For instance, 1-parameter subgroups of automorphisms of the algebra are equivalent to linear dynamical systems on the underlying vector space.

These dynamical systems should possess certain symmetries for the algebra to have a non-abelian structure. Hence, there is a close link between algebras, automorphic under the dynamics, and symmetries of dynamical systems. Linear dynamical systems find many physical applications. They form the basis of such an important field as quantum mechanics and underly the modern methods of integration of nonlinear dynamical systems.

In these notes, we study the general properties of Lie algebras that have a given linear dynamical system as its 1-parameter group of automorphisms. We call such algebras *automorphic algebras* of the dynamics. We study symmetries of the dynamics encoded by non-abelian automorphic algebras and their description in terms of a special limit transition in the space of Lie algebras along the dynamical trajectories. The latter relates automorphic algebras to the well-known Inönü-Wigner contraction and isospectral Lax representations. We give a few simple examples related to applications of this theory to classical matrix groups and nonlinear evolution equations. These connections make automorphic algebras a worth developing mathematical tool, useful in the theory of both dynamical systems and Lie algebras.

We assume that the reader is familiar with the basics of the Lie algebraic and group theory, for example, within the classical books [1–3].

1. Automorphic algebras and symmetries

Let V be a finite-dimensional complex vector space. Let a dynamic system G_t be given on V as a smooth 1-parameter group of linear transformations. In other words, G_t is a smooth representation of the additive group of real numbers on V:

$$G_t : R \to GL(V), \quad G_t G_s = G_{t+s},$$
$$G_0 = \mathrm{id}, \quad G_t^{-1} = G_{-t}.$$

The exponential map

$$G_t = e^{tM}, \quad M = \frac{d}{dt} G_t \Big|_{t=0}$$

identifies the trajectories

$$\{x(t) = G_t x(0)\} \subset V$$

with the solutions to the linear differential equation

$$\dot{x} = Mx, \quad x \in V, \tag{1}$$

to which G_t is a fundamental matrix. Note that G_t is always a subgroup of the general linear Lie group $GL(V)$ (of all transformations/automorphisms of V) and the operator M belongs to its Lie algebra $\mathfrak{gl}(V)$ (of all endomorphisms of V). The group G_t can be a 1-parameter subgroup of a smaller Lie group within $GL(V)$, and then M belongs to the relevant Lie algebra. Note also that the parameter t does not have to play the role of the time in the usual physical sense, but can be a more general evolution variable.

We additionally assume that the vector space V is equipped with a Lie algebraic structure with the bracket $[,]$ that satisfies the standard conditions of bilinearity, skew-symmetricity and the Jacobi identity. We denote the corresponding Lie algebra as $\mathfrak{a}$.

In this work we study the case where the group G_t acts on the algebra $\mathfrak{a}$ as a 1-parameter group of automorphisms, $G_t \in Aut(\mathfrak{a})$:

$$G_t[x, y] = [G_t x, G_t y] \quad \forall x, y \in \mathfrak{a}. \tag{2}$$

We call algebra $\mathfrak{a}$ *automorphic algebra* of the dynamical system G_t. By differentiation with respect to t, Eq. (2) is equivalent to the condition that the operator M is a derivation of $\mathfrak{a}$, $M \in \mathfrak{der}(\mathfrak{a})$:

$$M[x, y] = [Mx, y] + [x, My] \quad \forall x, y \in \mathfrak{a}. \tag{3}$$

Any dynamical system G_t is uniquely defined by its operator M. Hence, Eqs. (2), (3) identify all dynamical systems that have the same automorphic algebra $\mathfrak{a}$ with the Lie algebra $\mathfrak{der}(\mathfrak{a})$ of all derivations of $\mathfrak{a}$. The groups G_t span then the identity component $Aut(\mathfrak{a})_0$ of the group $Aut(\mathfrak{a})$ of all automorphisms of $\mathfrak{a}$. The group $Aut(\mathfrak{a})$ is a Lie group with the Lie algebra $\mathfrak{der}(\mathfrak{a})$. Due to the Jacobi identity, the algebra $\mathfrak{der}(\mathfrak{a})$ has a subalgebra (in fact, an ideal) $\mathfrak{ider}(\mathfrak{a})$ of inner derivations written via the adjoint representation of $\mathfrak{a}$ as $M = \mathrm{ad}(y) = [y, \cdot]$, $y \in \mathfrak{a}$. Each inner derivation generates a dynamical system G_t that belongs to the group of inner automorphisms $Inn(\mathfrak{a}) \subseteq Aut(\mathfrak{a})_0$. The corresponding Eq. (1) is of the form $\dot{x} = [y, x]$. For matrix/operator algebras this form corresponds to Lax equations. The latter are an important tool in the theory of nonlinear integrable systems and quantum mechanics [7–10]. We will encounter Lax equations again later when we consider semisimple automorphic subalgebras and limit transitions along the group trajectories. Note that the semisimple and nilpotent parts (in the sense of the Jordan decomposition) of any derivation M are also derivations of the same Lie algebra [2, 11].

A given dynamical system can have many automorphic algebras, not isomorphic to each other. For example, abelian algebras are automorphic for any dynamical system. Each automorphic algebra has the same 1-parameter group G_t of its automorphisms. Below we describe the basic general properties of automorphic algebras (i.e., the properties common for all automorphic algebras) of a given dynamical system and show that they encode an important information on its symmetries.

By an immediate observation, we come to the following consequence of Eq. (3) and the bilinearity of the bracket $[,]$.

Proposition 1. For any automorphic algebra, the bracket of any two solutions to Eq. (1) is again a solution to Eq. (1),

$$\dot{x} = Mx, \quad \dot{y} = My \quad \longrightarrow \quad \frac{d}{dt}[x, y] = M[x, y]. \tag{4}$$

In terms of Proposition 1, automorphic algebras impart an algebraic structure to the vector space of solutions to Eq. (1) (trajectories of the group G_t). Non-abelian automorphic algebras enable new solutions to Eq. (1) to be generated from known solutions that generically are not linear combinations of the latter.

It follows from the definition that any subalgebra $\mathfrak{a}_0$ of an automorphic algebra $\mathfrak{a}$ that is invariant under the dynamics, $M\mathfrak{a}_0 \subseteq \mathfrak{a}_0$, is an automorphic algebra of the restriction $G_t\big|_{\mathfrak{a}_0}$. Eq. (3) and the classical results of Refs. [12, 13] immediately imply the following statement.

Proposition 2. For any automorphic algebra $\mathfrak{a}$, its center, its derived algebra, its radical and its nilradical are invariant under the dynamics,

$$\begin{gathered} M\mathfrak{z}(\mathfrak{a}) \subseteq \mathfrak{z}(\mathfrak{a}), \quad M[\mathfrak{a},\mathfrak{a}] \subseteq [\mathfrak{a},\mathfrak{a}], \\ M\mathfrak{rad}(\mathfrak{a}) \subseteq \mathfrak{rad}(\mathfrak{a}), \quad M\mathfrak{nil}(\mathfrak{a}) \subseteq \mathfrak{nil}(\mathfrak{a}), \end{gathered} \tag{5}$$

and so are automorphic subalgebras of $\mathfrak{a}$.

In fact, those all are ideals of $\mathfrak{a}$ that are characteristic ideals, i.e., ideals invariant under any derivation M [12, 13]. The following embeddings take place $\mathfrak{z} \subseteq \mathfrak{nil} \subseteq \mathfrak{rad}$. In terms of Proposition 2, automorphic algebras enable to reduce the solutions to Eq. (1) into smaller invariant subspaces, giving the space of solutions additional structural properties.

More automorphic subalgebras can be constructed and the further analysis can be carried out in terms of the eigenspaces of the operator M, as shown below.

Let $\mathcal{E}$ be the set of (distinct) eigenvalues λ of the operator M and μ_λ denote the algebraic multiplicity of λ. Consider the decomposition of the vector space V into (generalised) eigenspaces of M

$$V = \bigoplus_{\lambda\in\mathcal{E}} V_\lambda, \quad V_\lambda = \{x \in V : (\lambda - M)^{\mu_\lambda} x = 0\}. \tag{6}$$

Eqs. (3), (6) imply the following classical result proved in Ref. [11].

Proposition 3. Any automorphic algebra admits the graded structure

$$\forall \lambda, \eta \in \mathcal{E} \quad [V_\lambda, V_\eta] \subseteq V_{\lambda+\eta}, \tag{7}$$

where we assume $V_\xi = 0$ if $\xi \notin \mathcal{E}$.

Multiplication of the operator M by any nonzero complex number $c = |c|e^{i\theta}$ homogeneously dilates and simultaneously rotates all the eigenvalues of M with respect to the origin on the complex plane. This however does not change automorphic algebras. Hence, the theory of automorphic algebras is invariant under the group of homotheties and rotations of the complex plane of eigenvalues with respect to the origin (the latter is a subgroup of the Möbius group of conformal transformations of the complex plane). In particular, it follows from Eq. (7) that, for any line that crosses the origin on the complex plane, the sets of eigenvalues that belong to this line, lie on one side of the line and lie on the opposite side of the line generate subalgebras of any automorphic algebra.

Further, we distinguish two qualitatively different situations, namely the cases where the operator M is non-degenerate, $0 \notin \mathcal{E}$, and where it is degenerate, $0 \in \mathcal{E}$. The non-degeneracy/degeneracy of M is equivalent to the non-existence/existence of non-trivial conservation laws ("integrals of motion"), i.e., non-zero elements $x \in V$ such that $Mx = 0, G_t x = x\, \forall t$.

The classical result proved in Ref. [14] implies that in the non-degenerate case all automorphic algebras are nilpotent. This is a consequence of Eq. (7) and the fact that the condition $\det M \neq 0$ implies the nilpotence of all adjoint representations $\mathrm{ad}(V_\lambda)$ and so (by a generalised Engel's theorem) the nilpotence of $\mathfrak{a}$ [14].

Proposition 4. If $0 \notin \mathcal{E}$ then all automorphic algebras are nilpotent.

Corrollary. The existence of non-nilpotent automorphic algebras implies the existence of nontrivial conservation laws.

A more detailed structure of nilpotent automorphic algebras in the non-degenerate case can be enlightened within the following definition.

Definition 1. An eigenvalue $\lambda \in \mathcal{E}$ is called *resonant* if $\lambda + \eta - \xi = 0$ for some $\eta,\, \xi \in \mathcal{E}$. Otherwise, λ is called *non-resonant.*

Proposition 5. For any automorphic algebra $\mathfrak{a}$, if λ is non-resonant then the relevant eigenspace belongs to the centre of $\mathfrak{a}$, $V_\lambda \subseteq \mathfrak{z}(\mathfrak{a})$. If all eigenvalues are non-resonant then all automorphic algebras are abelian.

Proof. Let λ be non-resonant. Then $\lambda + \eta - \xi \neq 0$ for all $\eta,\, \xi \in \mathcal{E}$. By Eq. (7), this implies $[V_\lambda, V_\eta] = 0$ for all $\eta \in \mathcal{E}$, as $\lambda + \eta$ cannot be an eigenvalue. Then, for any automorphic algebra $\mathfrak{a}$, we obtain $[V_\lambda, \mathfrak{a}] = 0$, i.e., $V_\lambda \subseteq \mathfrak{z}(\mathfrak{a})$. If all eigenvalues are non-resonant then all eigenspaces V_λ belong to the centre and $\mathfrak{a}$ is abelian. □

Since V_λ is invariant under G_t, it is an automorphic abelian subalgebra of $\mathfrak{a}$ for any non-resonant λ.

A simple example of applicability of Proposition 5 is the case where G_t is a 1-parameter subgroup of an irreducible representation of the (complexified) group $SO(3)$ on V. The group $SO(3)$ of the rotations of the Eucledian 3-space is an important group in physics, closely connected, for example, to the special unitary and special linear groups $SU(2), SL(2)$ as well as the Möbius group of conformal transformations of the complex plane. In this case, any operator M is the relevant representation of an element of the algebra $\mathfrak{so}(3)$. The latter can be treated as the algebra of quantum angular momentum operators [15]. Then there exists a basis of V such that

$$M = \alpha\, \mathrm{diag}\,(-S,\, -S+1,\, \ldots,\, S-1,\, S), \tag{8}$$

where α is some complex number, S is a positive integer or half-integer spin number that characterises the dimension of the representation, $\dim V = 2S+1$. For even-dimensional vector spaces V, the spin number S is half-integer corresponding to fermionic representations. For odd-dimensional V, the spin number S is integer and corresponds to bosonic representations. For fermionic representations, since S is half-integer, assuming $\alpha \neq 0$, it follows from Eq. (8) that the operator M is non-degenerate and the sum of any two eigenvalues of M is not an eigenvalue, i.e., by Definition 1, all eigenvalues are non-resonant. By Proposition 5, all automorphic algebras of any (nontrivial) dynamics generated by fermionic representations of $\mathfrak{so}(3)$ are abelian. We will return to this example later when we consider the bosonic case.

In terms of Proposition 5, for the existence of non-abelian automorphic algebras it is necessary that the resonance condition $\lambda + \eta - \xi = 0$ is satisfied for some $\lambda,\, \eta,\, \xi \in \mathcal{E}$. In the non-degenerate case, it means that the operator M has at least two distinct eigenvalues. Since 2-dimensional non-abelian algebras are non-nilpotent, we obtain then $\dim \mathfrak{a} = \dim V \geqslant 3$. The minimal example is the 3-dimensional Heisenberg algebra $\mathfrak{h}_3$

$$[v_\lambda, v_\eta] = v_\xi, \quad [v_\xi, v_\lambda] = [v_\xi, v_\eta] = 0.$$

Here λ, η are resonant and ξ is non-resonant, so $\operatorname{span}\{v_\xi\} = \mathfrak{z}$ and the centre is invariant under M in accordance with Proposition 2.

Nilpotent algebras are constructed as successive central extensions of abelian algebras, so any nilpotent algebra always has a nontrivial centre. Nilpotent algebras are solvable. All subalgebras and homomorphic images of nilpotent algebras are nilpotent. The Killing form on nilpotent algebras is zero. The adjoint representations of nilpotent algebras consist of nilpotent operators. Nilpotent algebras have outer automorphisms and outer derivations. So far, no general approach has been found to classification of nilpotent Lie algebras.

Let us now assume $0 \in \mathcal{E}$, i.e., M is a degenerate operator, $\det M = 0$. In this case, we can write $\lambda + 0 - \lambda = 0$, so, by Definition 1, all eigenvalues λ of the operator M are resonant. Eq. (7) immediately implies the following result.

Proposition 6. If $0 \in \mathcal{E}$ then the subspace V_0 is a nonzero subalgebra of any automorphic algebra that contains a nonzero subalgebra $\bar{V}_0 = \{x \in V_0 : Mx = 0\}$ of conservation laws. The adjoint representation of the subalgebra $\bar{V}_0$ acts on the space of solutions to Eq. (1): for any solution $y(t)$ to Eq. (1) within any automorphic algebra, the linear transformation

$$y'(t) = \operatorname{ad}(x) y(t) = [x, y(t)], \quad x \in \bar{V}_0, \tag{9}$$

gives again a solution to Eq. (1).

Since V_0, $\bar{V}_0$ are invariant under G_t, they are automorphic subalgebras. The operator M is nilpotent on V_0, so the restriction $G_t\big|_{V_0}$ is polynomial in t. Eq. (9) is a partial case of Eq. (4) that shows that, besides their conservative character, within automorphic algebras, nontrivial conservation laws of Eq. (1) play an important role in the structure of solutions.

To extend the result for the non-degenerate case to the degenerate case, it is natural to consider automorphic algebras as extensions of algebras that contain V_0 by nilpotent ideals. This can be done as follows.

Definition 2. The set $\mathcal{E}$ of eigenvalues of the operator M is called *split* if $\mathcal{E} = \mathcal{E}_0 \cup \bar{\mathcal{E}}$ with the properties:

i) $\mathcal{E}_0 \cap \bar{\mathcal{E}} = \emptyset$;

ii) $0 \in \mathcal{E}_0$;

iii) $\bar{\mathcal{E}} \neq \emptyset$;

iv) for any $\lambda_0,\ \eta_0 \in \mathcal{E}_0$, $\bar{\lambda},\ \bar{\eta} \in \bar{\mathcal{E}}$

$$\begin{aligned} &\text{either } \lambda_0 + \eta_0 \notin \mathcal{E} \text{ or } \lambda_0 + \eta_0 \in \mathcal{E}_0, \\ &\text{either } \bar{\lambda} + \bar{\eta} \notin \mathcal{E} \text{ or } \bar{\lambda} + \bar{\eta} \in \bar{\mathcal{E}}, \\ &\text{either } \lambda_0 + \bar{\lambda} \notin \mathcal{E} \text{ or } \lambda_0 + \bar{\lambda} \in \bar{\mathcal{E}}. \end{aligned} \tag{10}$$

Proposition 7. If the set $\mathcal{E}$ is split then any automorphic algebra is a semidirect sum

$$\mathfrak{a} = \mathfrak{a}_0 \dotplus \bar{\mathfrak{a}},$$

$$\mathfrak{a}_0 = \bigoplus_{\lambda \in \mathcal{E}_0} V_\lambda, \quad \bar{\mathfrak{a}} = \bigoplus_{\lambda \in \bar{\mathcal{E}}} V_\lambda,$$

where $\mathfrak{a}_0 \supseteq V_0$ is a subalgebra and $\bar{\mathfrak{a}}$ is a nilpotent ideal.

Proof. Indeed, by Eqs. (7), (10), we have

$$[\mathfrak{a}_0, \mathfrak{a}_0] \subseteq \mathfrak{a}_0, \quad [\bar{\mathfrak{a}}, \bar{\mathfrak{a}}] \subseteq \bar{\mathfrak{a}}, \quad [\mathfrak{a}_0, \bar{\mathfrak{a}}] \subseteq \bar{\mathfrak{a}}.$$

Hence, $\mathfrak{a}_0$ is a subalgebra and $\bar{\mathfrak{a}}$ is an ideal of any automorphic algebra. We have $0 \in \mathcal{E}_0$, so V_0 is a subalgebra of $\mathfrak{a}_0$. Since $0 \notin \bar{\mathcal{E}}$, by Proposition 4, the ideal $\bar{\mathfrak{a}}$ is nilpotent, as it is invariant under the operator M (forming then an automorphic subalgebra) where this operator is non-degenerate. The subalgebra $\mathfrak{a}_0$ acts on $\bar{\mathfrak{a}}$ by derivations, so the short exact sequence

$$\bar{\mathfrak{a}} \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{a}_0$$

defines a split extension of $\mathfrak{a}_0$, i.e., the semidirect sum $\mathfrak{a} = \mathfrak{a}_0 \dotplus \bar{\mathfrak{a}}$. □

In terms of Proposition 7, both $\mathfrak{a}_0$, $\bar{\mathfrak{a}}$ are invariant under the dynamics and so $\mathfrak{a}_0$, $\bar{\mathfrak{a}}$ are automorphic subalgebras. Proposition 5 can be applied then to the ideal $\bar{\mathfrak{a}}$ in terms of Definition 1 and the set $\bar{\mathcal{E}}$ — to specify the centre of $\bar{\mathfrak{a}}$. If several splittings of $\mathcal{E}$ exist, from the point of view of the Levi decomposition, in Proposition 7 the splitting with the minimal possible subset $\mathcal{E}_0$ should be chosen.

Since $\bar{\mathfrak{a}} \neq 0$, under the condition of Proposition 7, all automorphic algebras are non-semisimple. The important case where Proposition 7 is directly applicable is the *semidissipative* case

$$\forall \lambda \in \mathcal{E} \quad \operatorname{Re} \lambda \leqslant 0, \quad \exists \eta \in \mathcal{E} \quad \operatorname{Re} \eta < 0. \tag{11}$$

In this case, the set $\mathcal{E}$ is split into the subsets (remind $0 \in \mathcal{E}_0$)

$$\mathcal{E}_0 = \{\lambda \in \mathcal{E} : \operatorname{Re} \lambda = 0\}, \quad \bar{\mathcal{E}} = \{\lambda \in \mathcal{E} : \operatorname{Re} \lambda < 0\}.$$

This gives for any automorphic algebra the following semidirect sum of a subalgebra and a nilpotent ideal

$$\begin{aligned} &\mathfrak{a} = \mathfrak{a}_0 \dotplus \bar{\mathfrak{a}}, \\ &\mathfrak{a}_0 = \bigoplus_{\operatorname{Re} \lambda = 0} V_\lambda, \quad \bar{\mathfrak{a}} = \bigoplus_{\operatorname{Re} \lambda < 0} V_\lambda. \end{aligned} \tag{12}$$

Due to the invariance under rotations of eigenvalues with respect to the origin (see the comments after Proposition 3), the same situation occurs where the eigenvalues of the operator M are split into eigenvalues that belong to one side of a line that crosses the origin and eigenvalues that belong to this line.

We aim now to describe the situations where there exist non-solvable automorphic algebras, i.e., automorphic algebras with semisimple subalgebras. This is closely connected to projections of root systems of semisimple Lie algebras to the complex plane of eigenvalues of the operator M. The root systems of semisimple complex Lie algebras are fully classified [1–3].

Definition 3. A set of complex numbers $\rho(\mathfrak{g}) \subset C$ is called a *root projection* for a complex semisimple Lie algebra $\mathfrak{g}$ if there exists an element $m \in \mathfrak{h}$ of a maximal toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\rho(\mathfrak{g}) = \{\alpha(m) : \alpha \in \Phi\}$ where $\Phi \subset \mathfrak{h}^*$ is the set of roots of $\mathfrak{g}$ corresponding to $\mathfrak{h}$.

Proposition 8. Let a root projection of a semisimple complex Lie algebra $\mathfrak{g}$ exist such that $\rho(\mathfrak{g}) \subset \mathcal{E}$. Let $\mu'_\lambda \geqslant \bar{\mu}_\lambda$ if $0 \neq \lambda \in \rho(\mathfrak{g})$. Let $\mu'_0 \geqslant r + \bar{\mu}_0$ if $0 \in \rho(\mathfrak{g})$ where $r = \operatorname{rank}\mathfrak{g}$. Here μ'_λ is the geometric multiplicity of λ in $\mathcal{E}$ and $\bar{\mu}_\lambda$ is the multiplicity of λ in $\rho(\mathfrak{g})$. Then there exists an automorphic algebra $\mathfrak{a}$ with a semisimple subalgebra $\mathfrak{a}_0$ isomorphic to $\mathfrak{g}$.

Proof. Consider the vector space

$$\mathfrak{a}_0 = \bar{V}_0 \oplus \bigoplus_{\lambda \in \rho(\mathfrak{g})} \bar{V}_\lambda,$$

$$\bar{V}_0 = \bigoplus_{k=1}^{r} v_0^{(k)} \subseteq V_0, \quad \bar{V}_\lambda = \bigoplus_{q=1}^{\bar{\mu}_\lambda} v_\lambda^{(q)} \subseteq V_\lambda,$$

where $v_0^{(k)}$, $v_\lambda^{(q)}$ are eigenvectors of the operator M corresponding respectively to the eigenvalues 0 and λ. Such eigenvectors exist by the conditions imposed on the multiplicities of the eigenvalues. By Eq. (7) and the root space decomposition of $\mathfrak{g}$, the vector space $\mathfrak{a}_0$ is a semisimple automorphic subalgebra of an automorphic algebra $\mathfrak{a}$ with the restriction $M_0 \equiv M\big|_{\mathfrak{a}_0} = \operatorname{ad}(m)$. To get an automorphic algebra $\mathfrak{a}$ on the full space V, it is sufficient to consider the trivial extension of the subalgebra $\mathfrak{a}_0$ by the abelian subalgebra $h = V \setminus \mathfrak{a}_0$. □

In terms of Proposition 8, the semisimple subalgebra $\mathfrak{a}_0$ is invariant under the dynamics, so it is an automorphic subalgebra. The set $\rho(\mathfrak{g})$ is centrally symmetric with respect to the origin on the complex plane. This implies $\operatorname{Tr} M_0 = 0$. Hence, for the restriction $G_t^0 = G_t\big|_{\mathfrak{a}_0}$ we obtain $\det G_t^0 = \exp(t \operatorname{Tr} M_0) = 1\ \forall t$ and G_t^0 belongs then to the special linear Lie group $SL(\mathfrak{a}_0)$. As a result, G_t^0 preserves the volume and orientation of the vector space $\mathfrak{a}_0$. Note also that the restriction M_0 is always a semisimple operator and an inner derivation of $\mathfrak{a}_0$. The restriction of Eq. (1) on $\mathfrak{a}_0$ is of the Lax type $\dot{x} = [m, x]$.

The simplest example of applicability of Proposition 8 is the case $0, \pm\alpha \in \mathcal{E}$ with an arbitrary complex number α. In this case, there exists an automorphic subalgebra isomorphic to $A_1 = \mathfrak{sl}(2) = \mathfrak{so}(3)$. For instance, getting back to Eq. (8), this situation is realised for bosonic representations of the algebra $\mathfrak{so}(3)$ where the spin number S is integer and so any operator M has the eigenvalues $0, \pm\alpha$. By Proposition 8, unlike the case of fermionic representations where all automorphic algebras are abelian (see the comments after Proposition 5), any dynamics generated by bosonic representations of $\mathfrak{so}(3)$ has an automorphic subalgebra isomorphic to $\mathfrak{so}(3)$ and so has a non-solvable automorphic algebra. By Propositions 5 and 8, a strict algebraic difference exists between fermions and bosons in terms of automorphic algebras.

The situation is somewhat similar in the case where $G_t \subset SO(N)$. For the even series of the orthogonal groups $D_n = SO(2n)$ and $M \neq 0$ all automorphic algebras are nilpotent, while for the odd series $B_n = SO(2n+1)$ automorphic subalgebras exist, isomorphic to $\mathfrak{sl}(2)$.

The lowest-dimensional bosonic case $S = 1$ corresponds to the standard 3-dimensional representation of $\mathfrak{so}(3)$. In this case, as a consequence of the fact $\{0, \pm\alpha\} = \mathcal{E}$ and Eq. (7), the dynamics G_t generated by any operator M from this representation (plane uniform rotations around a fixed coordinate line) has three non-abelian automorphic algebras that are not isomorphic to each other:

$$\begin{aligned}
\mathfrak{so}(3) = \mathfrak{sl}(2): &\quad [v_0, v_\pm] = \pm v_\pm, \quad [v_+, v_-] = v_0,\\
\mathfrak{h}_3: &\quad [v_0, v_\pm] = 0, \quad [v_+, v_-] = v_0,\\
\mathfrak{e}(2): &\quad [v_0, v_\pm] = \pm v_\pm, \quad [v_+, v_-] = 0.
\end{aligned}$$

Here v_0, $v_\pm$ are the eigenvectors of M corresponding to the eigenvalues $0, \pm\alpha$. These algebras are respectively simple ($\mathfrak{so}(3)$), nilpotent (the Heisenberg algebra $\mathfrak{h}_3$) and non-nilpotent solvable (the Euclidean algebra $\mathfrak{e}(2)$).

At the end of this section, we point out that the spectral problem (6) for the operator M on V generates a symmetric spectral problem for the operator $\operatorname{ad}(M)$ on $\mathfrak{gl}(V)$ in terms of the adjoint representations of automorphic algebras.

Proposition 9. For any automorphic algebra $\mathfrak{a}$, for each $\lambda \in \mathcal{E}$ and each $v \in V_\lambda$, the adjoint representation $\operatorname{ad}(v)$ of the element $v \in \mathfrak{a}$ satisfies the spectral problem

$$(\lambda - \operatorname{ad}(M))^{\mu_\lambda} \operatorname{ad}(v) = 0. \tag{13}$$

Proof. Indeed, in terms of the operators $\operatorname{ad}(x) = [x, \cdot]$, Eq. (3) is written as

$$\operatorname{ad}(Mx) - [M, \operatorname{ad}(x)] = \operatorname{ad}(Mx) - \operatorname{ad}(M)\operatorname{ad}(x) = 0.$$

For $v \in V_\lambda$, we have $(\lambda - M)^{\mu_\lambda} v = 0$. Utilizing the Jordan form of M on V_λ, we can choose a basis $\{v_1, \ldots, v_{\mu_\lambda}\}$ in V_λ such that

$$M v_k = \lambda v_k + \sum_{s=1}^{k-1} c_{ks} v_s, \quad k = 1, \ldots, \mu_\lambda$$

for some complex constants c_{ks}. This implies

$$\operatorname{ad}(M v_k) = \lambda \operatorname{ad}(v_k) + \sum_{s=1}^{k-1} c_{ks} \operatorname{ad}(v_s)$$

and we come to the fact that the operators $\operatorname{ad}(v_k)$ all satisfy Eq. (13). Precisely,

$$(\lambda - \operatorname{ad}(M))^k \operatorname{ad}(v_k) = 0, \quad k = 1, \ldots, \mu_\lambda. \quad \square$$

In terms of decomposition (6) and Proposition 9, if the operator M is semisimple on V_λ then $c_{ks} = 0$ and both spectral problems (6), (13) are split on V_λ:

$$(\lambda - M) v_k = 0, \quad (\lambda - \operatorname{ad}(M)) \operatorname{ad}(v_k) = 0. \tag{14}$$

We have $M, \operatorname{ad}(v_k) \in \mathfrak{Der}(\mathfrak{a}) \subset \mathfrak{gl}(V)$, so Proposition 9 and Eqs. (13), (14) reduce the procedure of finding automorphic algebras to the usual linear algebra.

2. Automorphic algebras and generalised Inönü-Wigner contractions

In this section we show that automorphic algebras of a given dynamical system G_t can be produced from non-automorphic algebras by a special limit transition along the trajectories of G_t. This limit procedure generalises the well-known Inönü-Wigner contraction [16] that finds a variety of physical applications [10, 17–23].

Let an algebra Lie $\mathfrak{a}$ with a bracket $[,]$ (generically non-automorphic for G_t) be given on the vector space V. Consider the bilinear operation on V

$$[x, y]_t = G_{-t}[G_t x, G_t y], \quad t \in R, \quad x, y \in V. \quad (15)$$

For all t, the bracket $[,]_t$ inherits the bilinearity, skew-symmetricity and Jacobi identity of the bracket $[,]$ of the algebra $\mathfrak{a}$. Hence, each bracket $[,]_t$ defines a Lie algebra $\mathfrak{a}_t$ on V.

Proposition 10. Let for all $x, y \in V$ there exist the finite limit (in the standard topology of the vector space $V = C^n$)

$$[x, y]' = \lim_{t\to+\infty} [x, y]_t. \quad (16)$$

Then the limit algebra $\mathfrak{a}'$ with the bracket $[,]'$ is automorphic for G_t.

Proof. By differentiation of Eq. (15) with respect to t, we get for all t, x, y

$$\frac{d}{dt}[x, y]_t = -M[x, y]_t + [Mx, y]_t + [x, My]_t. \quad (17)$$

The existence of the finite limit (16) implies both left-hand and right-hand sides of Eq. (17) to vanish at $t \to +\infty$. The operator M becomes a derivation of the limit algebra $\mathfrak{a}'$. The latter is then automorphic for the dynamical system G_t. □

Due to the relation $[G_t x, G_t y] = G_t[x, y]_t$, for each finite t, the intermediate algebra $\mathfrak{a}_t$ is isomorphic to $\mathfrak{a}$. If $\mathfrak{a}$ is automorphic for G_t then $[x, y]_t = [x, y]$ is independent of t and the intermediate algebras $\mathfrak{a}_t$ and the limiting algebra $\mathfrak{a}'$ all coincide with $\mathfrak{a}$. Otherwise, the limit $\mathfrak{a}'$ is a Lie algebra that is (in general) not isomorphic to $\mathfrak{a}$, although $\mathfrak{a}$ and $\mathfrak{a}'$ have the same underlying vector space V.

In terms of the decomposition (6) into eigenspaces of M, for the limit of Eq. (16) to exist, it is sufficient to generalise the graded structure of Eq. (7) to

$$[V_\lambda, V_\eta] \subset \bigoplus V_\xi, \quad \xi = \lambda + \eta \text{ or } \operatorname{Re}\xi > \operatorname{Re}\lambda + \operatorname{Re}\eta. \quad (18)$$

The limit (16) transforms the grading (18) to the grading (7).

The limit transition (15), (16) enables to describe automorphic algebras of a dynamical system G_t in a self-consistent way, as limit cases of any algebras, satisfying Eq. (18), along the trajectories of the group G_t itself. Similarly to Eq. (16), we can consider the limit

$$[x, y]'_- = \lim_{t\to-\infty} [x, y]_t. \quad (19)$$

Provided the latter exists, we again come to a new Lie algebra $\mathfrak{a}'_-$ that is automorphic for G_t. The two limits (16), (19) are mutually connected by inversion of the signs of the eigenvalues of M. If both limits (16), (19) simultaneously exist then $\mathfrak{a}' = \mathfrak{a}'_-$.

For semidissipative dynamical systems (11), the procedure (15), (16) is equivalent to the Inönü-Wigner contraction. In this case, the vector space V shrinks (contracts) along the trajectories of G_t. As per Proposition 7 and Eq. (12), limit algebras of the Inönü-Wigner contraction are always non-semisimple. They are split extensions of the subalgebra $\mathfrak{a}_0$ spanned by the eigenspaces of M corresponding to purely imaginary eigenvalues by the nilpotent ideal $\bar{\mathfrak{a}}$ spanned by the eigenspaces of M corresponding to eigenvalues with negative real parts. The restriction of Eq. (15) onto $\mathfrak{a}_0$ is either compact for all t or has terms that polynomially grow with t, so for the limit (16) to exist, the bracket $[x, y]_t$ should be independent of t for $x, y \in \mathfrak{a}_0$. Then the limit algebra keeps the initial bracket on $\mathfrak{a}_0$. As a result, there exists the homomorphism

$$G' : \mathfrak{a}' \to \mathfrak{a}, \quad \ker G' = \bar{\mathfrak{a}}, \quad \operatorname{fix} G' = \mathfrak{a}_0 \quad (20)$$

that realises the aforementioned split extension (the short exact sequence)

$$\bar{\mathfrak{a}} \longrightarrow \mathfrak{a}' \longrightarrow \mathfrak{a}_0.$$

For example, in the original setting [16, 17], the Inönü-Wigner contraction corresponds to the case

$$Mx = 0, \quad My = \lambda y, \quad \operatorname{Re}\lambda < 0, \quad G_t x = x, \quad G_t y = e^{\lambda t} y, \quad x \in \mathfrak{a}_0, \quad y \in h, \quad (21)$$

where $\mathfrak{a}_0 \subset \mathfrak{a}$ is a subalgebra, h is the complementary subspace. In the limit $t \to +\infty$, according to Eqs. (15), (16), (18), we come to the new Lie bracket on V that keeps $\mathfrak{a}_0$ as a subalgebra and makes h an abelian ideal,

$$[\mathfrak{a}_0, \mathfrak{a}_0]' = [\mathfrak{a}_0, \mathfrak{a}_0] \subseteq \mathfrak{a}_0, \quad [\mathfrak{a}_0, h]' \subseteq h, \quad [h, h]' = 0. \quad (22)$$

Eqs. (15), (16) generalise the Inönü-Wigner procedure to any, not only semidissipative dynamics satisfying Eq. (18). Unlike Eq. (20), we do not require the limit algebra to be a split extension of a nonzero algebra. The limit algebra $\mathfrak{a}'$ can be semisimple in certain cases where the initial algebra $\mathfrak{a}$ is semisimple.

To give a simple example, qualitatively different from Eqs. (21), (22), consider the 3-dimensional algebra $\mathfrak{a}$ spanned by vectors v_-, v_0, v_+ with

$$Mv_\xi = \xi\lambda v_\xi, \quad \xi = -, 0, +, \quad \operatorname{Re}\lambda > 0, \quad [v_-, v_0] = 2v_- + \alpha v_0 + \beta v_+, \quad [v_+, v_-] = v_0 + \alpha v_+, \quad [v_0, v_+] = 2v_+, \quad (23)$$

where α, β are arbitrary complex numbers. The bracket $[,]$ satisfies the Jacobi identity and so defines a Lie algebra. Taking the limit (16), we come to the new bracket

$$[v_-, v_0]' = 2v_-, \quad [v_+, v_-]' = v_0, \quad [v_0, v_+]' = 2v_+ \quad (24)$$

that is the bracket of the algebra $\mathfrak{a}' = \mathfrak{sl}(2)$ that is a simple Lie algebra. The aforedefined operator M is a derivation of the new algebra, so $\mathfrak{a}'$ is automorphic for G_t. This is however not the case for the initial algebra $\mathfrak{a}$ unless $\alpha = \beta = 0$.

This example illustrates also Proposition 8. Here $M = \mathrm{ad}(\lambda v_0/2)$ (v_0 spans the maximal toral subalgebra) with the eigenvalue 0 and the two non-zero eigenvalues $\pm\lambda$ generated by the set of roots for $\mathfrak{sl}(2)$.

In the example (23), (24), the limit algebra $\mathfrak{a}' = \mathfrak{sl}(2)$ is isomorphic to the initial one $\mathfrak{a}$. Examples of non-dissipative dynamics generating non-isomorphic limit algebras also can be easily given. For example, it is sufficient to modify Eq. (23) as

$$\begin{gathered} M v_\xi = \lambda_\xi v_\xi, \quad \xi = -,\, 0,\, +, \\ \lambda_0 = 0, \quad \mathrm{Re}\,\lambda_+ > 0, \\ \mathrm{Re}\,\lambda_- < -\mathrm{Re}\,\lambda_+ < 0, \\ [v_-, v_0] = 2v_- + \alpha v_0 + \beta v_+, \\ [v_+, v_-] = v_0 + \alpha v_+, \quad [v_0, v_+] = 2v_+. \end{gathered} \tag{25}$$

After the limit (16), we obtain

$$\begin{gathered} [v_-, v_0]' = 2v_-, \quad [v_+, v_-]' = 0, \\ [v_0, v_+]' = 2v_+. \end{gathered} \tag{26}$$

The operator M is a derivation of $\mathfrak{a}'$, so the limit algebra is indeed automorphic. Here the initial algebra is isomorphic to $\mathfrak{sl}(2)$ while the limit algebra (isomorphic to the Lie algebra $\mathfrak{e}(2)$ of the Euclidean group $E(2)$) is solvable and so non-isomorphic to $\mathfrak{sl}(2)$.

Example (25), (26) also illustrates Proposition 7. Choosing $\mathcal{E}_0 = \{0\}$, $\bar{\mathcal{E}} = \{\lambda_+,\, \lambda_-\}$, we see that the set $\mathcal{E}$ is split. Hence, indeed, the subspace $\bar{\mathfrak{a}} = \mathrm{span}\,(v_-, v_+)$ is a nilpotent (abelian in this case) ideal in $\mathfrak{a}'$. This subspace coincides with the derived algebra, $\bar{\mathfrak{a}} = [\mathfrak{a}', \mathfrak{a}']$. This illustrates Proposition 2, as $\bar{\mathfrak{a}}$ is indeed invariant under G_t. In accordance with Propositions 6, 7, the 1-dimensional subspace spanned by v_0 is a subalgebra that contains conservation laws.

As the final result of this study, we point out that the spectral problem (14) for semisimple operators M can be linked to the limit procedure (15), (16) via a Lax representation in $\mathfrak{gl}(V)$. In fact, in terms of the adjoint representation $\mathrm{ad}_t(v_\lambda)$ in the intermediate algebras $\mathfrak{a}_t$, Eq. (17) is recast as

$$\frac{d}{dt}\,\mathrm{ad}_t(v_\lambda) = (\lambda - \mathrm{ad}(M))\mathrm{ad}_t(v_\lambda). \tag{27}$$

We used the fact that M is semisimple, so the eigenspace V_λ is split into a set of eigenvectors v_λ, thus splitting the spectral problem (13) into Eq. (14). Eq. (27) easily implies the following result.

Proposition 11. The operator

$$L_\lambda(t) = e^{-\lambda t}\mathrm{ad}_t(v_\lambda) \in \mathfrak{gl}(V) \tag{28}$$

satisfies the isospectral Lax representation

$$\frac{d}{dt}\,L_\lambda(t) = [L_\lambda(t), M]. \tag{29}$$

In particular, the eigenvalues of $L_\lambda(t)$ and analytical functions of them are conservation laws of Eq. (29).

In terms of proposition 11, the limit (15), (16) is equivalent to the limit along the trajectories of Eq. (27)

$$\mathrm{ad}_0(v_\lambda) \to \mathrm{ad}'(v_\lambda), \quad t \to +\infty,$$

where $\mathrm{ad}_0(v_\lambda)$, $\mathrm{ad}'(v_\lambda)$ are the adjoint representations in the initial and the limit algebras $\mathfrak{a}$, $\mathfrak{a}'$. The trajectories are found by the transformation (28) and the Lax representation of Eq. (29). It is worth mentioning that the adjoint representations $\mathrm{ad}_t(v_0)$ corresponding to conservation laws of Eq. (1) directly satisfy the Lax representation of Eq. (29) without the transformation of Eq. (28). Proposition 11 implies the following statement.

Proposition 12. Let the finite limit (15), (16) exist. Then, for $\mathrm{Re}\,\lambda \geqslant 0$, $\lambda \neq 0$, the operators $\mathrm{ad}_t(v_\lambda)$ are nilpotent for all t. For $\mathrm{Re}\,\lambda < 0$ the limit operator $\mathrm{ad}'(v_\lambda)$ is nilpotent. The eigenvalues of the operator $\mathrm{ad}_t(v_0)$ that corresponds to $\lambda = 0$ (and so all their analytical functions) are conservation laws of Eqs. (27), (29).

Proof. By Proposition 11, the eigenvalues of the operator $L_\lambda(t)$ of Eq. (28) are conservation laws of Eq. (29). Then any eigenvalue $\alpha_t(\lambda)$ of the operator $\mathrm{ad}_t(v_\lambda)$ has the form $\alpha_t(\lambda) = e^{\lambda t}\alpha_0(\lambda)$, where $\alpha_0(\lambda)$ is an eigenvalue of the operator $\mathrm{ad}_0(v_\lambda)$ of the initial algebra. Hence, if $\mathrm{Re}\,\lambda \geqslant 0$, $\lambda \neq 0$, for the limit (15), (16) to exist, it is necessary $\alpha_0(\lambda) = 0$, so $\alpha_t(\lambda) = 0$ for all t, i.e., $\mathrm{ad}_t(v_\lambda)$ should be nilpotent for all t. If $\mathrm{Re}\,\lambda < 0$ then $\alpha_t(\lambda) \to 0$, $t \to +\infty$, i.e., the limit operator $\mathrm{ad}'(v_\lambda)$ is nilpotent. For $\lambda = 0$, we have $L_0(t) = \mathrm{ad}_t(v_0)$, so the eigenvalues of $\mathrm{ad}_t(v_0)$ are conservation laws of Eqs. (27), (29). □

It follows from Proposition 12 that, for any $\lambda \neq 0$, the adjoint representation $\mathrm{ad}'(v_\lambda)$ in the limit algebra $\mathfrak{a}'$ is a nilpotent operator. In fact, it follows from Eq. (7) that, for any automorphic algebra, for all $v \in V_\lambda$ with $\lambda \neq 0$, the operator $\mathrm{ad}(v)$ is nilpotent (the condition of semisimplicity of M can be lifted). Remarkably, the conservation laws v_0 of Eq. (1) on automorphic algebras on V generate conservation laws $\mathrm{Tr}\,[\mathrm{ad}_t(v_0)^m]$ of Eq. (29) on the algebra $\mathfrak{gl}(V)$.

Propositions 10-12 along with Proposition 9 illustrate the remarkable algebraic role of the limit procedure (15), (16) for description of adjoint representations of automorphic algebras.

Conclusion

Automorphic Lie algebras of linear dynamical systems have been introduced as Lie algebraic structures on the space of their trajectories. We have formulated the basic general properties of automorphic Lie algebras of a given dynamical system in terms of the eigenspace decomposition of the dynamics. We have pointed out the symmetries that are encoded by the presence of non-abelian automorphic algebras. In particular, non-nilpotent automorphic algebras are related to conservation laws of the dynamics. In the presence of a semisimple automorphic subalgebra, there is a natural correspondence between the set of roots related to the subalgebra to the set of eigenvalues of the dynamical system. We have shown that automorphic algebras can be found by a limit transition along the trajectories of the dynamics, a procedure that generalises the well-known Inönü-Wigner contraction. We have demonstrated that, in terms of the adjoint representation, the limit transition is naturally reduced to an isospectral Lax representation. We have given simple examples related to applications of the developed theory to classical matrix groups. This suggests that automorphic algebras are worth developing tool in the theory of both dynamical systems and Lie algebras.

Inönü-Wigner contractions, in the dynamical setting of Eqs. (15), (16), have been applied before to dissipative dynamical systems [20, 22, 23]. Other dynamical deformations of Lie algebras, both related and unrelated to Inönü-Wigner contractions, have also been discussed [24, 25]. We are unaware of whether the automorphic character of limit algebras has ever been noticed. We are also unaware of an earlier use of non-dissipative dynamical systems for constructing non-isomorphic algebras on the same vector space. As far as we know, the link of the limit transition of the Inönü-Wigner type to the Lax representations has not been made before.

As direct physical applications of the methodology developed in this work, we would expect first of all cases where the groups G_t have additional special properties: for example, belong to various physically important symmetry groups and their representations. Among them, we can find classical and relativistic mechanics (the classical matrix groups, the Galilean, Lorentz and Poincaré groups [26]) and quantum applications such as, for example, Lindblad equations of open quantum dynamics (completely positive quantum semigroups [27]). Considering finite and discrete groups that generate discrete dynamical systems would be curious as well (for example, within the theory of Ref. [5]).

It would be interesting, to our mind, to consider also infinite-dimensional underlying vector spaces V, especially functional spaces, or finite-dimensional Lie algebras over functional rings. In the latter cases, it might be expected that the linear systems (1), the graded structures (7) and the Lax representations (29) are related to some integrable nonlinear evolution equations of mathematical physics [7–10]. According to Proposition 4, for an open set of dynamical systems, all automorphic algebras are nilpotent. Nilpotent algebras play an important role in the representation theory, especially in the orbit method and geometric quantisation [28]. It would be curious to build links of these modern theories to the theory of automorphic algebras we developed.

Some of the results we obtained can be reformulated for arbitrary (not necessarily Lie) algebras, making this subject useful in a wider algebraic context.

To give one simple example in relation to the last two paragraphs, consider the vector space V of smooth complex functions $x : \Xi \to C$ given on a smooth real manifold Ξ of a dimension n with (local) coordinates $\xi = (\xi_1, \ldots, \xi_n)$. Let a smooth vector field $F(\xi) = (F_1(\xi), \ldots, F_n(\xi))$ be given on Ξ and let Eq. (1) be generated by the operator M of differentiation along the field F:

$$\dot{x} = Mx \equiv \langle F, \nabla x \rangle = \sum_{k=1}^{n} F_k \frac{\partial x}{\partial \xi_k}. \tag{30}$$

Then the solutions $x(t, \xi)$ are given by evolution of the initial value $x(0, \xi) = x_0(\xi)$ along the flow on Ξ generated by the vector field F: $x(t, \xi) = x_0(\eta(t, \xi))$ where

$$\dot{\eta} = F(\eta), \quad \eta(0, \xi) = \xi \tag{31}$$

(the group G_t is a realisation of such evolution). The operator M is a differentiation "from the left", so M is a derivation of the associative algebra on V generated by the usual product $x, y \to xy$. This algebra is automorphic for the dynamics of Eq. (30). In particular, the product of any two solutions $x(t, \xi)y(t, \xi)$ is again a solution. The conservation laws $Mx(\xi) = 0$ of Eq. (30) are in a one-to-one correspondence with the conservation laws $\langle F, \nabla x \rangle = 0$ of Eq. (31). In some cases, this observation helps to find conservation laws for the nonlinear dynamics of Eq. (31) from the linear dynamics of Eq. (30).

The classical case is the Hamiltonian dynamics where the manifold Ξ is even-dimensional, $n = 2m$, and F is a Hamiltonian vector field:

$$F(\xi) = J\nabla h(\xi), \quad J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

Here $h(\xi) \in V$ is the Hamiltonian, J is the matrix of a symplectic bilinear form on Ξ (I_m is the $m \times m$ unit matrix). The Hamiltonian $h(\xi)$ is always a conservation law for Eq. (31). In fact, $Mx = \{h, x\}$, so $Mh = \{h, h\} = 0$ where the Poisson bracket

$$\{x, y\} = \langle J\nabla x, \nabla y \rangle$$

defines a Lie algebraic structure on the functional vector space V. In some cases, the condition $\{h, x\}$=0 provides additional conservation laws x of Eq. (31). Along with the associative algebra generated by the usual product, the Lie algebra with the Poisson bracket is also automorphic for the dynamics of Eq. (30). This imparts the relevant Lie algebraic structure to the space of solutions to Eq. (30). Here $M = \mathrm{ad}(h)$ is an inner derivation, so in the Hamiltonian case Eq. (30) $\dot{x} = \{h, x\}$ is of the Lax type. The symplectic structure makes the manifold Ξ a symplectic manifold. Any symplectic manifold can be realised as an orbit of the coadjoint representation of some Lie group [28]. Extensions related to partial differential equations and quantum mechanics are possible [9, 28].

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