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Stueckelberg particle in the uniform electric field, solutions with cylindrical symmetry

A. V. Ivashkevich[1], P. O. Sachenok[2], V. M. Red'kov[1]

[1]B. I. Stepanov Institute of Physics of the National Academy of Sciences of Belarus, Minsk, Belarus
[2]Mozyr State Pedagogical University named after I. P. Shamyakin, Mozyr, Belarus

ivashkevich.alina@yandex.by
polinasacenok@gmail.com
v.redkov@ifanbet.bas-net.by

Abstract

In the present paper, the system of 11 equations for massive Stueckelberg particle is studied in presence of the external uniform electric field. We apply covariant formalism according to the general tetrad approach by Tetrode-Weyl-Fock-Ivanenko specified for cylindrical coordinates. After separating the variables, we derive the system of the first-order differential equations in partial derivatives with respect to coordinates (r, z). To resolve this system, we apply the Fedorov-Gronskiy method, thereby we consider the 11-dimensional spin operator and find on this base three projective operators, which permit us to expand the complete wave function in the sum of three parts. Besides, according to the general method, dependence of each projective constituent on the variable r should be determined by only one function. Also, in accordance with the general method we impose the first-order constraints which permit us to transform all differential equations in partial derivatives with respect to coordinates (r, z) into the system of 11 first-order ordinary differential equations in the variable z. The last system is solved in terms of confluent hypergeometric functions. In total, four independent types of solutions have been constructed, in contrast to the case of the ordinary spin 1 particle described by Daffin-Kemmer equation when only three types of solutions are possible.

Keywords:

Stueckelberg particle, tetrad formalism, cylindrical symmetry, external electric field, separation of the variables, differential equations in partial derivatives, exact solutions

Частица Штюкельберга в электрическом поле, решения с цилиндрической симметрией

А. В. Ивашкевич[1], П. О. Саченок[2], В. М. Редьков[1]

[1]Институт физики имени Б. И. Степанова Национальной академии наук Беларуси, г. Минск, Беларусь
[2]Мозырский государственный педагогический университет имени И. П. Шамякина, г. Мозырь, Беларусь

ivashkevich.alina@yandex.by
polinasacenok@gmail.com
v.redkov@ifanbet.bas-net.by

Аннотация

В настоящей работе система 11 уравнений для массивной частицы Штюкельберга исследуется в присутствии внешнего однородного электрического поля. Применяется тетрадный формализм, согласно методу Тетрода-Вейля-Фока-Иваненко. Используются цилиндрические координаты и соответствующая диагональная тетрада. Разделив переменные, получили систему дифференциальных уравнений первого порядка в частных производных по координатам (r, z). Для решения этой системы применяется метод Федорова-Гронского, согласно которому на основе 11-мерного оператора спина введены три проективных оператора, позволяющие разложить полную волновую функцию в сумму трех частей. Согласно общему методу, зависимость каждой проективной составляющей от переменной r должна определяться только одной функцией. Также используются дифференциальные ограничения первого порядка, совместимые с системой уравнений и позволяющие преобразовать все уравнения в частных производных по координатам (r, z) в обыкновенные дифференциальные уравнения по переменной z. Последняя система решена в терминах вырожденных гипергеометрических функций. Построены четыре независимые решения, в отличие от случая обычной частицы со спином 1, описываемой уравнением Даффина-Кемера, когда возможны только три решения.

Ключевые слова:

частица Штюкельберга, тетрадный формализм, цилиндрическая симметрия, внешнее электрическое поле, разделение переменных, уравнения в частных производных, точные решения

1. The basic equation

The initial Stueckelberg system [1–5] of equations for a massive particle in presence of external electromagnetic fields is

$$-D^a\Psi_a - \mu\Psi = 0,$$

$$D_a\Psi + D^b\Psi_{ab} - \mu\Psi_a = 0,$$

$$D_a\Psi_b - D_b\Psi_a - \mu\Psi_{ab} = 0,$$

where $D_a = \partial_a + ieA_a$. As the wave function, we will use the 11-dimensional column

$$\Phi(x) = (\Psi; \Psi_0, \Psi_1, \Psi_2, \Psi_3;$$

$$\Psi_{01}, \Psi_{02}, \Psi_{03}, \Psi_{23}, \Psi_{31}, \Psi_{12})^t = (H, H_1, H_2)^t,$$

where ^{t} denotes transposition. The above system can be presented in the block form

$$D_aG^aH_1 + \mu H = 0,$$

$$\Delta^aD_aH + K^aD_aH_2 - \mu H_1 = 0, \qquad (1)$$

$$D_aL^aH_1 - \mu H_2 = 0,$$

or differently

$$(-D_a\Gamma^a - \mu)\Phi = 0,$$

$$\Gamma^a = \begin{pmatrix} 0 & -G^a & 0 \\ \Delta^a & 0 & K^a \\ 0 & L^a & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} H \\ H_1 \\ H_2 \end{pmatrix}. \qquad (2)$$

All blocks were defined in [3–5]. This matrix equation for Stueckelberg particle can be extended to the Riemannian space-time in accordance with the known tetrad procedure

$$\left[\Gamma^\alpha(x)\left(\frac{\partial}{\partial x^\alpha} + \sum_\alpha(x) - ieA_\alpha\right) - \mu\right]\Psi(x) = 0. \qquad (3)$$

Local matrices $\Gamma^\alpha(x)$ are determined through the tetrads

$$\Gamma^\alpha(x) = e^\alpha_{(a)}(x)\Gamma^a =$$

$$= \begin{pmatrix} 0 & -G^ae^\alpha_{(a)} & 0 \\ \Delta^ae^\alpha_{(a)} & 0 & K^ae^\alpha_{(a)} \\ 0 & L^ae^\alpha_{(a)} & 0 \end{pmatrix}. \qquad (4)$$

The connection $\Sigma_\alpha(x)$ is defined by the formulas

$$j^{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & j_1^{ab} & 0 \\ 0 & 0 & j_2^{ab} \end{pmatrix},$$

$$\Sigma_\alpha(x) = \frac{1}{2}j^{ab}e^\beta_{(a)}(x)e_{(b)\beta;\alpha}(x), \qquad (5)$$

$$\Sigma_1(x) = \frac{1}{2}j^{ab}_{(1)}e^\beta_a(x)e_{(b)\beta;\alpha}(x),$$

$$\Sigma_2(x) = \frac{1}{2}j^{ab}_{(2)}e^\beta_a(x)e_{(b)\beta;\alpha}(x),$$

where $j^{ab}_{(1)}$ and $j^{ab}_{(2)}$ designate generators for vector $\Psi_k(x)$ and antisymmetric tensor $\Psi_{[mn]}(x)$, respectively. Equation (3) may be presented with the use of the Ricci rotation coefficients

$$\left[\Gamma^c\left(e^\alpha_{(c)}\frac{\partial}{\partial x^\alpha} + \frac{1}{2}j^{ab}\gamma_{abc} - ieA_c\right) - \mu\right]\Psi(x) = 0. \qquad (6)$$

In more detailed form, Eq. (6) reads

$$-G^c\left(e^\alpha_{(c)}\partial_\alpha + j^{ab}_{(1)}\frac{1}{2}\gamma_{abc} - ieA_c\right)H_1 - \mu H = 0,$$

$$\Delta^c\left(e^\alpha_{(c)}\partial_\alpha - ieA_c\right)H +$$

$$+K^c\left(e^\alpha_{(c)}\partial_\alpha + j^{ab}_{(2)}\frac{1}{2}\gamma_{abc} - ieA_c\right)H_2 - \mu H_1 = 0, \qquad (7)$$

$$L^c\left(e^\alpha_{(c)}\partial_\alpha + j^{ab}_{(1)}\frac{1}{2}\gamma_{abc} - ieA_c\right)H_1 - \mu H_2 = 0.$$

Let us consider the Stueckelberg equation in presence of the uniform electric field. In cylindrical coordinates and corresponding diagonal tetrad

$$x^\alpha = (t, r, \phi, z),$$

$$dS^2 = dt^2 - dr^2 - r^2d\phi^2 - dz^2, \; A_0 = -Ez,$$

the above equation takes the form (let $eE \Rightarrow E$):

$$\left[\Gamma^0\left(\frac{\partial}{\partial t} + iEz\right) + \Gamma^1\frac{\partial}{\partial r} + \right.$$

$$\left. + \Gamma^2\frac{\partial_\phi + j^{12}}{r} + \Gamma^3\frac{\partial}{\partial z} - \mu\right]\Psi = 0. \qquad (8)$$

In block form, it reads

$$\left[-G^0\left(\frac{\partial}{\partial t} + iEz\right) - G^1\frac{\partial}{\partial r} - \right.$$

$$\left. -G^2\frac{1}{r}\left(\frac{\partial}{\partial\phi} + j_1^{12}\right) - G^3\frac{\partial}{\partial z}\right]H_1 - \mu H = 0,$$

$$\left[\Delta^0\left(\frac{\partial}{\partial t} + iEz\right) + \Delta^1\frac{\partial}{\partial r} + \Delta^2\frac{1}{r}\partial_\phi + \Delta^3\frac{\partial}{\partial z}\right]H +$$

$$+\left[K^0\left(\frac{\partial}{\partial t} + iEz\right) + K^1\frac{\partial}{\partial r} + \right. \qquad (9)$$

$$\left. + K^2\frac{\partial_\phi + j_2^{12}}{r} + K^3\frac{\partial}{\partial z}\right]H_2 = \mu H_1,$$

$$\left[L^0\frac{\partial}{\partial t} + L^1\frac{\partial}{\partial r} + L^2\frac{\partial_\phi + j_1^{12}}{r} + L^3\frac{\partial}{\partial z}\right]H_1 = \mu H_2.$$

In the following, it will be convenient to apply the cyclic basis, in which the third projection of the spin is diagonal (see details in [3–5]).

2. Separation of the variables

We apply the following substitution for the wave function in cyclic basis

$$\bar{\Psi} = e^{-i\epsilon t} e^{im\phi} \begin{pmatrix} \bar{H}(r,z) \\ \bar{H}_1(r,z) \\ \bar{H}_2(r,z) \end{pmatrix}, \quad \bar{H} = h(r,z),$$

$$\bar{H}_1 = \begin{pmatrix} h_0(r,z) \\ h_1(r,z) \\ h_2(r,z) \\ h_3(r,z) \end{pmatrix}, \quad \bar{H}_2 = \begin{pmatrix} E_i(r,z) \\ B_i(r,z) \end{pmatrix}. \qquad (10)$$

Then Eqs. (9) read

$$\Big[+i(\epsilon - Ez)G^0 - G^1 \frac{d}{dr} -$$

$$-G^2 \frac{1}{r}\left(im + j_1^{12}\right) - \frac{d}{dz} G^3 \Big] H_1 = \mu H,$$

$$\Big[-i(\epsilon - Ez) m \Delta^0 + \Delta^1 \frac{d}{dr} + \frac{im}{r}\Delta^2 + \frac{d}{dz}\Delta^3 \Big] H +$$

$$+ \Big[-i(\epsilon - Ez)\bar{K}^0 + K^1 \frac{d}{dr} +$$

$$+K^2 \frac{im + j_2^{12}}{r} + \frac{d}{dz} K^3 \Big] H_2 = \mu H_1,$$

$$\Big[-i(\epsilon - Ez) L^0 + L^1 \frac{d}{dr} +$$

$$+L^2 \frac{im + j_1^{12}}{r} + \frac{d}{dz} L^3 \Big] H_1 = \mu H_2.$$

After simple calculation, we obtain the system of 11 equations. With the use of the shortening notations

$$a_m = \frac{d}{dr} + \frac{m}{r}, \quad a_{m+1} = \frac{d}{dr} + \frac{m+1}{r},$$
$$b_m = \frac{d}{dr} - \frac{m}{r}, \quad b_{m-1} = \frac{d}{dr} - \frac{m-1}{r}, \qquad (11)$$

it reads

$$i(\epsilon - Ez)h_0 + \frac{d}{dz}h_2 - b_{m-1}h_1 + a_{m+1}h_3 = \mu h,$$

$$-i(\epsilon - Ez)h - \frac{d}{dz}E_2 + b_{m-1}E_1 - a_{m+1}E_3 = \mu h_0,$$

$$-a_m h + a_{m+1}B_2 - \frac{d}{dz}B_3 + i(\epsilon - Ez)E_1 = \mu h_1,$$

$$\frac{d}{dz}h + i(\epsilon - Ez)E_2 - a_{m+1}B_1 - b_{m-1}B_3 = \mu h_2,$$

$$b_m h + b_m B_2 + \frac{d}{dz}B_1 + i(\epsilon - Ez)E_3 = \mu h_3,$$

$$a_m h_0 - i(\epsilon - Ez)h_1 = \mu E_1,$$

$$-\frac{d}{dz}h_0 - i(\epsilon - Ez)h_2 = \mu E_2,$$

$$-b_m h_0 - i(\epsilon - Ez)h_3 = \mu E_3,$$

$$-b_m h_2 + \frac{d}{dz}h_3 = \mu B_1,$$

$$b_{m-1}h_1 + a_{m+1}h_3 = \mu B_2, \quad -\frac{d}{dz}h_1 - a_m h_2 = \mu B_3.$$

3. The Fedorov-Gronskiy method

To resolve the last system, we will implement the Fedorov-Gronskiy method [6]. To this end, let us consider the 11-dimensional spin operator $Y = -i\bar{J}^{12}$. We readily verify that it satisfies the minimal equation $Y(Y-1)(Y+1) = 0$. This permits us to introduce three projective operators

$$P_1 = \frac{1}{2}Y(Y-1), \quad P_2 = \frac{1}{2}Y(Y+1),$$
$$P_3 = 1 - Y^2, \quad P_0 + P_{+1} + P_{-1} = 1. \qquad (12)$$

Therefore, the complete wave function may be decomposed into the sum of three parts

$$\Psi = \Psi_0 + \Psi_{+1} + \Psi_{-1},$$
$$\Psi_\sigma = P_\sigma \Psi, \quad \sigma = 0, +1, -1. \qquad (13)$$

We can readily find an explicit formula of them. Besides, according to the Fedorov-Gronskiy method, dependence of each projective constituent on the variable r should be determined by only one function

$$\Psi_1(r,z) = \big(0, 0, h_1(z), 0, 0, E_1(z),$$
$$0, 0, 0, 0, B_3(z)\big)^t f_1(r),$$
$$\Psi_2(r,z) = \big(0, 0, 0, 0, h_3(z), 0, 0, E_3(z),$$
$$B_1(z), 0, 0\big)^t f_2(r),$$
$$\Psi_3(r,z) = \big(h_1(z), h_0(z), 0, h_2(z),$$
$$0, 0, E_2(z), 0, 0, B_2(z), 0\big)^t f_3(r). \qquad (14)$$

Acting by projective operators on the above system of 11 equations $P_i(A_{11\times 11}\Psi) = 0$, we get three subsystems. Besides, in accordance with the general method, we should impose the first-order constraints which permit us to transform all differential equations in partial derivatives with respect to coordinates (r,z) into the system of ordinary differential equations of the variable z

$$P_1$$

$$-a_m f_3(r)h(z) + a_m f_3(r)B_2(z) - f_1(r)\frac{d}{dz}B_3(z) +$$
$$+i(\epsilon - Ez)f_1(r)E_1(z) = \mu f_1(r)h_1(z) \Rightarrow$$
$$a_m f_3(r) = C_1 f_1(r),$$
$$a_m f_3(r)h_0(z) - i(\epsilon - Ez)f_1(r)h_1(z) =$$
$$= \mu f_1(r)E_1(z) \Rightarrow a_m f_3(r) = C_1 f_1(r),$$
$$-f_1(r)\frac{d}{dz}h_1(z) - a_m f_3(r)h_2(z) =$$
$$= \mu f_1(r)B_3(z) \Rightarrow a_m f_3(r) = C_1 f_1(r);$$

$$P_2$$

$$b_m f_3(r)h(z) + b_m f_3(r)B_2(z) + f_2(r)\frac{d}{dz}B_1(z)+$$
$$+i(\epsilon - Ez)f_2(r)E_3(z) = \mu f_2(r)h_3(z) \Rightarrow$$
$$b_m f_3(r) = C_2 f_2(r),$$
$$-b_m f_3(r)h_0(z) - i(\epsilon - Ez)f_2(r)h_3(z) =$$
$$= \mu f_2(r)E_3(z) \Rightarrow b_m f_3(r) = C_2 f_2(r),$$
$$-b_m f_3(r)h_2(z) + f_2(r)\frac{d}{dz}h_3(z) =$$
$$= \mu f_2(r)B_1(z) \Rightarrow b_m f_3(r) = C_2 f_2(r);$$
$$P_3$$
$$-i(\epsilon - Ez)f_3(r)h_0(z) - f_3(r)\frac{d}{dz}h_2(z)+$$
$$+b_{m-1}f_1(r)h_1(z) - b_{m-1}f_1(r)h_3(z) =$$
$$= \mu f_3(r)h(z) \Rightarrow b_{m-1}f_1(r) = C_3 f_3(r),$$
$$-i(\epsilon - Ez)f_3(r)h(z) - f_3(r)\frac{d}{dz}E_2(z)+$$
$$b_{m-1}f_1(r)E_1(z) - a_{m+1}f_2(r)E_3(z) = \mu f_3(r)h_0(z)$$
$$\Rightarrow b_{m-1}f_1(r) = C_3 f_3(r), \quad a_{m+1}f_2(r) = C_4 f_3(r),$$
$$f_3(r)\frac{d}{dz}h(z) + i(\epsilon - Ez)f_3(r)E_2(z)-$$
$$-a_{m+1}f_2(r)B_1(z) - b_{m-1}f_1(r)B_3(z) = \mu f_3(r)h_2(z)$$
$$\Rightarrow b_{m-1}f_1(r) = C_3 f_3(r), \quad a_{m+1}f_2(r) = C_4 f_3(r),$$
$$-f_3(r)\frac{d}{dz}h_0(z) - i\epsilon f_3(r)h_2(z) = \mu f_3(r)E_2(z),$$
$$b_{m-1}f_1(r)h_1(z) + a_{m+1}f_2(r)h_3(z) = \mu f_3(r)B_2(z)$$
$$\Rightarrow b_{m-1}f_1(r) = C_3 f_3(r), \quad a_{m+1}f_2(r) = C_4 f_3(r).$$

Thus, we get the following system

$$-C_1 h + C_1 B_2 - \frac{d}{dz}B_3 + i(\epsilon - Ez)E_1 = \mu h_1,$$
$$C_1 h_0 - i(\epsilon - Ez)h_1 = \mu E_1, \quad -\frac{d}{dz}h_1 - C_1 h_2 = \mu B_3,$$
$$C_2 h + C_2 B_2 + \frac{d}{dz}B_1 + i(\epsilon - Ez)E_3 = \mu h_3,$$
$$-C_2 h_0 - i(\epsilon - Ez)h_3 = \mu E_3,$$
$$-C_2 h_2 + \frac{d}{dz}h_3 = \mu B_1,$$
$$-i(\epsilon - Ez)h_0 - \frac{d}{dz}h_2 + C_3 h_1 - C_3 h_3 = \mu h,$$
$$-i(\epsilon - Ez)h - \frac{d}{dz}E_2 + C_3 E_1 - C_4 E_3 = \mu h_0,$$
$$\frac{d}{dz}h + i(\epsilon - Ez)E_2 - C_4 B_1 - C_3 B_3 = \mu h_2,$$
$$-\frac{d}{dz}h_0 - i(\epsilon - Ez)h_2 = \mu E_2, \quad C_3 h_1 + C_4 h_3 = \mu B_2,$$

and the constraints

$$b_{m-1}f_1(r) = C_3 f_3(r), \quad a_m f_3(r) = C_1 f_1(r),$$
$$a_{m+1}f_2(r) = C_4 f_3(r), \quad b_m f_3(r) = C_2 f_2(r). \qquad (15)$$

Eqs. (15) transform into equations for separate functions

$$b_{m-1}a_m f_3(r) = C_1 C_3 f_3(r),$$
$$a_m b_{m-1} f_1(r) = C_1 C_3 f_1(r),$$
$$a_{m+1}b_m f_3(r) = C_2 C_4 f_3(r), \qquad (16)$$
$$b_m a_{m+1} f_2(r) = C_2 C_4 f_2(r).$$

Evidently, within each pair we can assume $C_3 = C_1, C_4 = C_2$. Therefore, the above differential conditions and the second-order equations take on the form

$$b_{m-1}f_1(r) = C_1 f_3(r), \quad a_m f_3(r) = C_1 f_1(r),$$
$$a_{m+1}f_2(r) = C_2 f_3(r), \quad b_m f_3(r) = C_2 f_2(r); \qquad (17)$$

$$[b_{m-1}a_m - C_1^2]f_3(r) = 0,$$
$$[a_m b_{m-1} - C_1^2]f_1(r) = 0,$$
$$f_3(r) = 0, \quad [b_m a_{m+1} - C_2^2]f_2(r) = 0. \qquad (18)$$

Explicitly, Eqs. (18) are red as

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - C_1^2\right) f_3(r) = 0,$$
$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-1)^2}{r^2} - C_1^2\right) f_1(r) = 0,$$
$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - C_2^2\right) f_3(r) = 0,$$
$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1)^2}{r^2} - C_2^2\right) f_2(r) = 0.$$

So we get the following constraint $C_3^2 = C_2^2 = C_1^2 = C^2$, and only three different equations

$$1 \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-1)^2}{r^2} - C^2\right) f_1(r) = 0,$$
$$2 \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1)^2}{r^2} - C^2\right) f_2(r) = 0, \qquad (19)$$
$$3 \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - C^2\right) f_3(r) = 0.$$

They are solved in Bessel functions. More details on the parameter C^2 are given later. The meaning of parameter C^2 may be understood if we turn to the Klein-Fock-Gordon equation in cylindrical coordinates in presence of the uniform electric field

$$\left[\frac{d^2}{dz^2} + (\epsilon - Ez)^2 + \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dz} - \frac{m^2}{r^2} - \mu^2\right] \times$$
$$\times e^{i\epsilon t}e^{im\phi}R(r)Z(z) = 0.$$

The variables are separated as follows

$$\left[\frac{d^2}{dz^2} + (\epsilon - Ez)^2 - \mu^2 + \lambda\right] Z(z) = 0,$$

$$\left[\frac{d^2}{dz^2}+\frac{1}{r}\frac{d}{dz}-\frac{m^2}{r^2}-\lambda\right]R(r)=0,$$

so $C^2=\lambda$ is the separation constant associated with the cylindrical coordinate system (see (19)).

4. Solving the equations in the variable z

Below we will take into account the identities $C_1=C_2=C_3=C$. We should solve the system of equations in the variable z:

$$-Ch+CB_2-\frac{d}{dz}B_3+i(\epsilon-Ez)E_1=\mu h_1,$$
$$Ch_0-i(\epsilon-Ez)h_1=\mu E_1,\ -\frac{d}{dz}h_1-Ch_2=\mu B_3,$$
$$Ch+CB_2+\frac{d}{dz}B_1+i(\epsilon-Ez)E_3=\mu h_3,$$
$$-Ch_0-i(\epsilon-Ez)h_3=\mu E_3,\ -Ch_2+\frac{d}{dz}h_3=\mu B_1,$$
$$-i(\epsilon-Ez)h_0-\frac{d}{dz}h_2+Ch_1-Ch_3=\mu h, \quad (20)$$
$$-i(\epsilon-Ez)h-\frac{d}{dz}E_2+CE_1-CE_3=\mu h_0,$$
$$\frac{d}{dz}h+i(\epsilon-Ez)E_2-CB_1-CB_3=\mu h_2,$$
$$-\frac{d}{dz}h_0-i(\epsilon-Ez)h_2=\mu E_2,\ Ch_1+Ch_3=\mu B_2.$$

First, we resolve the subsystem of 6 equations

$$-i(\epsilon-Ez)h-\frac{d}{dz}E_2+CE_1-CE_3=\mu h_0,$$
$$\frac{d}{dz}h+i(\epsilon-Ez)E_2-CB_1-CB_3=\mu h_2,$$
$$Ch_0-i(\epsilon-Ez)h_1=\mu E_1,$$
$$-Ch_0-i(\epsilon-Ez)h_3=\mu E_3, \quad (21)$$
$$-Ch_2+\frac{d}{dz}h_3=\mu B_1,\quad -\frac{d}{dz}h_1-Ch_2=\mu B_3;$$

as algebraic one with respect to the variables h_0,h_2,E_1,E_3,B_1,B_3. This results in (let $d_z=\frac{d}{dz}$)

$$h_0=\frac{d_zE_2\mu-i(Ch_1-Ch_3+h\mu)(Ez-\epsilon)}{2C^2-\mu^2},$$
$$h_2=\frac{-d_z(Ch_1-Ch_3+h\mu)+iE_2\mu(Ez-\epsilon)}{2C^2-\mu^2},$$
$$E_1=\frac{1}{2C^2\mu-\mu^3}\Big(Cd_zE_2\mu+i(h_1(C-\mu)(C+\mu)+$$
$$+C(Ch_3-h\mu))(Ez-\epsilon)\Big), \quad (22)$$
$$E_3=\frac{1}{\mu^3-2C^2\mu}\Big(Cd_zE_2\mu-i\Big(C^2h_1+Ch\mu+$$
$$+h_3(C-\mu)(C+\mu)\Big)(Ez-\epsilon)\Big),$$
$$B_1=\frac{1}{\mu^3-2C^2\mu}\Big(-d_z\Big(C^2h_1+Ch\mu+$$
$$+h_3(C-\mu)(C+\mu)\Big)+iCE_2\mu(Ez-\epsilon)\Big),$$
$$B_3=\frac{1}{\mu^3-2C^2\mu}\Big(d_z\left(h_1(C-\mu)(C+\mu)+\right.$$
$$\left.+C(Ch_3-h\mu)\right)+iCE_2\mu(Ez-\epsilon)\Big).$$

Now substitute these expressions into remaining 5 equations

$$-i(\epsilon-Ez)h_0-\frac{d}{dz}h_2+Ch_1-Ch_3=\mu h,$$
$$-\frac{d}{dz}h_0-i(\epsilon-Ez)h_2=\mu E_2,$$
$$Ch_1+Ch_3=\mu B_2, \quad (23)$$
$$-Ch+CB_2-\frac{d}{dz}B_3+i(\epsilon-Ez)E_1=\mu h_1,$$
$$Ch+CB_2+\frac{d}{dz}B_1+i(\epsilon-Ez)E_3=\mu h_3.$$

As a result, we obtain

1 $$\frac{d_z^2h\mu}{2C^2-\mu^2}+\frac{Cd_z^2h_1}{2C^2-\mu^2}-\frac{Cd_z^2h_3}{2C^2-\mu^2}+$$
$$+\mu\left(\frac{(\epsilon-Ez)^2}{2C^2-\mu^2}-1\right)h+\left(\frac{C(\epsilon-Ez)^2}{2C^2-\mu^2}+C\right)h_1+$$
$$+Ch_3\left(\frac{(\epsilon-Ez)^2}{\mu^2-2C^2}-1\right)=0;$$

2 $$\frac{C^2d_z^2h_3}{2C^2\mu-\mu^2}+\frac{d_z^2h_1(\mu^2-C^2)}{\mu^3-2C^2\mu}+$$
$$+\frac{Cd_z^2h}{\mu^2-2C^2}+B_2C-\frac{C^2h_3(\epsilon-Ez)^2}{\mu^3-2C^2\mu}+$$
$$+\left(-\frac{(\mu^2-C^2)(\epsilon-Ez)^2}{2C^2\mu-\mu^3}-\mu\right)h_1+$$
$$+C\left(\frac{(\epsilon-Ez)^2}{\mu^2-2C^2}-1\right)h=0;$$

3 $$\frac{C^2d_z^2h_1}{2C^2\mu-\mu^3}+\frac{d_z^2h_3(\mu^2-C^2)}{\mu^3-2C^2\mu}+$$
$$+\frac{Cd_z^2h}{2C^2-\mu^2}+B_2C+\frac{C^2h_1(\epsilon-Ez)^2}{2C^2\mu-\mu^3}+$$
$$+h_3\left(-\frac{(\mu^2-C^2)(\epsilon-Ez)^2}{2C^2\mu-\mu^3}-\mu\right)+$$
$$+\left(\frac{C(\epsilon-Ez)^2}{2C^2-\mu^2}+C\right)h=0;$$

4 $$\left[\frac{d^2}{dz^2}+(\epsilon-Ez)^2-\mu^2-2C^2\right]E_2=0;$$

5 $$-\mu B_2+Ch_1+Ch_3=0.$$

With the use of equation 5, from equations 2 and 3 we can eliminate the variable B_2. In this way we obtain the system of 3 equations for variables h, h_1, h_3

$$
\begin{aligned}
1 \qquad & \mu^2(d_z^2 - 2C^2 + \mu^2 + (\epsilon - Ez)^2)h + \\
& + C\mu(d_z^2 + 2C^2 - \mu^2 + (\epsilon - Ez)^2)h_1 - \\
& - C\mu(d_z^2 + 2C^2 - \mu^2 + (\epsilon - Ez)^2)h_3 = 0, \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
2 \qquad & C\mu(d_z^2 + 2C^2 - \mu^2 + (\epsilon - Ez)^2)h + \\
& + (\mu^2 - C^2)(d_z^2 + 2C^2 - \mu^2 + (\epsilon - Ez)^2)h_1 - \\
& - C^2(d_z^2 + 2C^2 - \mu^2 + (\epsilon - Ez)^2)h_3 = 0, \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
3 \qquad & C\mu(d_z^2 + 2C^2 - \mu^2 + (\epsilon - Ez)^2)h + \\
& + C^2(d_z^2 + 2C^2 - \mu^2 + (\epsilon - Ez)^2)h_1 - \\
& - (\mu^2 - C^2)(d_z^2 + 2C^2 - \mu^2 + (\epsilon - Ez)^2)h_3 = 0. \qquad (26)
\end{aligned}
$$

The structure of these equations may be presented shortly as follows

$$
\begin{aligned}
1 \qquad & A_1 h'' + B_1 h + C_1 h_1'' + \\
& + D_1 h_1 + M_1 h_3'' + N_1 h_3 = 0, \\
2 \qquad & A_2 h'' + B_2 h + C_2 h_1'' + \\
& + D_2 h_1 + M_2 h_3'' + N_2 h_3 = 0, \\
3 \qquad & A_3 h'' + B_3 h + C_3 h_1'' + \\
& D_3 h_1 + M_3 h_3'' + N_3 h_3 = 0. \qquad (27)
\end{aligned}
$$

We will combine these equations in three different ways.

The first variant is

$$
\begin{aligned}
(aA_1 + bA_2 + cA_3)^{-1}h'' + (aB_1 + bB_2 + cB_3)h + \\
+ (aC_1 + bC_2 + cC_3)^{-0}h_1'' + \\
+ (aD_1 + bD_2 + cD_3)h_1 + \\
+ (aM_1 + bM_2 + cM_3)^{-0}h_3'' + \\
+ (aN_1 + bN_2 + cN_3)h_3 = 0.
\end{aligned}
$$

This results in

$$
\begin{aligned}
h'' + (aB_1 + bB_2 + cB_3)h + (aD_1 + bD_2 + cD_3)h_1 + \\
+ (aN_1 + bN_2 + cN_3)h_3 = 0,
\end{aligned}
$$

where a, b, c obey the linear system

$$
\begin{aligned}
aA_1 + bA_2 + cA_3 = 1, \\
aC_1 + bC_2 + cC_3 = 0, \qquad (28) \\
aM_1 + bM_2 + cM_3 = 0;
\end{aligned}
$$

its solution is

$$
a = \frac{1}{\mu^2 - 2C^2}, \quad b = \frac{C}{2C^2\mu - \mu^3}, \quad c = \frac{C}{2C^2\mu - \mu^3}.
$$

The second variant is

$$
\begin{aligned}
(aA_1 + bA_2 + cA_3)^{-0}h'' + (aB_1 + bB_2 + cB_3)h + \\
+ (aC_1 + bC_2 + cC_3)^{-1}h_1'' + \\
+ (aD_1 + bD_2 + cD_3)h_1 + \\
+ (aM_1 + bM_2 + cM_3)^{-0}h_3'' + \\
+ (aN_1 + bN_2 + cN_3)h_3 = 0.
\end{aligned}
$$

This results in

$$
\begin{aligned}
h_1'' + (aB_1 + bB_2 + cB_3)h + (aD_1 + bD_2 + cD_3)h_1 + \\
+ (aN_1 + bN_2 + cN_3)h_3 = 0,
\end{aligned}
$$

where a, b, c obey the linear system

$$
\begin{aligned}
aA_1 + bA_2 + cA_3 = 0, \\
aC_1 + bC_2 + cC_3 = 1, \qquad (29) \\
aM_1 + bM_2 + cM_3 = 0;
\end{aligned}
$$

its solution is

$$
a = \frac{C}{2C^2\mu - \mu^3}, \quad b = \frac{1}{\mu^2 - 2C^2}, \quad c = 0.
$$

The third variant is

$$
\begin{aligned}
(aA_1 + bA_2 + cA_3)^{-0}h'' + (aB_1 + bB_2 + cB_3)h + \\
+ (aC_1 + bC_2 + cC_3)^{-0}h_1'' + \\
+ (aD_1 + bD_2 + cD_3)h_1 + \\
+ (aM_1 + bM_2 + cM_3)^{-1}h_3'' + \\
+ (aN_1 + bN_2 + cN_3)h_3 = 0.
\end{aligned}
$$

This results in

$$
\begin{aligned}
h_3'' + (aB_1 + bB_2 + cB_3)h + (aD_1 + bD_2 + cD_3)h_1 + \\
+ (aN_1 + bN_2 + cN_3)h_3 = 0,
\end{aligned}
$$

where a, b, c obey the linear system

$$
\begin{aligned}
aA_1 + bA_2 + cA_3 = 0, \\
aC_1 + bC_2 + cC_3 = 0, \qquad (30) \\
aM_1 + bM_2 + cM_3 = 1;
\end{aligned}
$$

its solution is

$$
a = \frac{C}{\mu^3 - 2C^2\mu}, \quad b = 0, \quad c = \frac{1}{2C^2 - \mu^2}.
$$

So after this transformation we get three second-order separate equations

$$
\begin{aligned}
\frac{d^2}{dz^2}h + (2C^2 + \mu^2 + (\epsilon - Ez)^2)h = 0, \\
\frac{d^2}{dz^2}h_1 + (2C^2 - \mu^2 + (\epsilon - Ez)^2)h_1 - 2Ch\mu = 0, \\
\frac{d^2}{dz^2}h_3 + (2C^2 - \mu^2 + (\epsilon - Ez)^2)h_3 + 2Ch\mu = 0. \qquad (31)
\end{aligned}
$$

Let us introduce new variables

$$
H = h_1 + h_3, \quad G = h_1 - h_3. \qquad (32)
$$

Then instead of (31) we can obtain one separate equation

$$
\left[\frac{d^2}{dz^2} + 2C^2 - \mu^2 + (\epsilon - Ez)^2\right] H = 0 \qquad (33)
$$

and one subsystem

$$\left[\frac{d^2}{dz^2} + 2C^2 + \mu^+ + (\epsilon - Ez)^2\right] h = 0,$$

$$\left[\frac{d^2}{dz^2} + 2C^2 + \mu^2 + (\epsilon - Ez)^2\right] G -$$

$$-2\mu^2 G - 4\mu C h = 0.$$

The last subsystem can be presented in the matrix form

$$D \begin{pmatrix} h \\ G \end{pmatrix} = 2\mu \begin{pmatrix} 0 & 0 \\ 2C & \mu \end{pmatrix} \begin{pmatrix} h \\ G \end{pmatrix},$$

$$D\Psi = 2\mu A \Psi. \tag{34}$$

Let us find transformation which diagonalizes the mixing matrix A

$$\bar{\Psi} = S\Psi, \quad D\bar{\Psi} = 2\mu(SAS^{-1})\bar{\Psi}, \quad \bar{\Psi} = \begin{pmatrix} \bar{h} \\ \bar{G} \end{pmatrix}.$$

For transformation matrix S we derive the following equations

$$SA = \bar{A}S, \quad \bar{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2C & \mu \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix},$$

whence it follows

$$s_{12} 2C = \lambda_1 s_{11}, \quad s_{12}\mu = \lambda_1 s_{12},$$

$$\begin{pmatrix} \lambda_1 & -2C \\ 0 & (\lambda_1 - \mu) \end{pmatrix} \begin{pmatrix} s_{11} \\ s_{12} \end{pmatrix} = 0,$$

$$s_{22} 2C = \lambda_2 s_{21}, \quad s_{22}\mu = \lambda_2 s_{22},$$

$$\begin{pmatrix} \lambda_2 & -2C \\ 0 & (\lambda_2 - \mu) \end{pmatrix} \begin{pmatrix} s_{21} \\ s_{22} \end{pmatrix} = 0.$$

The first row is specified by relations $\lambda_1 = 0, s_{12} = 0, s_{11} = 1$; the second row is specified as $\lambda_2 = \mu, s_{22} = 1, s_{21} = 2C/\mu$. Thus, the needed transformation matrix S is

$$S = \begin{pmatrix} 1 & 0 \\ \frac{2C}{\mu} & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{2C}{\mu} & 1 \end{pmatrix}. \tag{35}$$

Therefore, we derive three separate equations:

$$\left[\frac{d^2}{dz^2} + 2C^2 + \mu^2 + (\epsilon - Ez)^2\right] \bar{h} = 0, \tag{36}$$

$$\left[\frac{d^2}{dz^2} + 2C^2 - \mu^2 + (\epsilon - Ez)^2\right] \bar{G} = 0, \tag{37}$$

$$\left[\frac{d^2}{dz^2} + 2C^2 - \mu^2 + (\epsilon - Ez)^2\right] \bar{H} = 0, \tag{38}$$

where

$$\bar{h} = h, \quad \bar{H} = H = h_1 + h_3,$$

$$G = h_1 - h_3, \quad \bar{G} = \frac{2C}{\mu} h + h_1 - h_3.$$

Besides we should remember the existence of the fourth independent equation for the variable E_2:

$$\left[\frac{d^2}{dz^2} - 2C^2 - \mu^2 + (\epsilon - Ez)^2\right] E_2 = 0. \tag{39}$$

So, in total, four independent types of solutions exist for Stueckelberg particle in the external uniform electric field, in contrast to the ordinary spin 1 particle described by the Daffin-Kemmer equation when only three independent solutions are possible. All four equations (36)–(39) have the same mathematical structure. In the papers [7, 8], solutions for equation of the form (39) were constructed in terms of the confluent hypergeometric functions.

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