Stückelberg particle in external magnetic field. Nonrelativistic approximation. Exact solutions

Частица Штюкельберга во внешнем магнитном поле. Нерелятивистское приближение. Точные решения

E.M. Ovsiyuk[1], A.P. Safronov[1], A.V. Ivashkevich[2], O.A. Semenyuk[3]

[1]Mozyr State Pedagogical University
named after I.P. Shamyakin,
Mozyr, Belarus
[2]B.I. Stepanov Institute of Physics
of the National Academy of Sciences of Belarus,
Minsk, Belarus
[3]Brest State University named after A.S. Pushkin
Brest, Belarus

e.ovsiyuk@mail.ru
safronov_mspu@mail.ru
ivashkevich.alina@yandex.by
olya.vasilyuk.97@yandex.by

Е.М. Овсиюк[1], А.П. Сафронов[1], А.В. Ивашкевич[2], О.А. Семенюк[3]

[1]Мозырский государственный педагогический
университет имени И.П. Шамякина,
г. Мозырь, Беларусь
[2]Институт физики имени Б.И. Степанова
Национальной академии наук Беларуси,
г. Минск, Беларусь
[3]Брестский государственный университет имени А.С. Пушкина,
г. Брест, Беларусь

e.ovsiyuk@mail.ru
safronov_mspu@mail.ru
ivashkevich.alina@yandex.by
olya.vasilyuk.97@yandex.by

Abstract

The Stückelberg equation for a particle with two spin states, $S = 1$ and $S = 0$, is studied in the presence of an external uniform magnetic field. In relativistic case, the particle is described by an 11-component wave function. On the solutions of the equation, the operators of energy, the third projection of the total angular momentum, and the third projection of the linear momentum along the direction of the magnetic field are diagonalized. After separation of variables, we derive a system for 11 functions depending on one variable. We perform the nonrelativistic approximation in this system. For this we apply the known method of deriving nonrelativistic equations from relativistic ones, which is based on projective operators related to the matrix Γ_0 of the relativistic equation. The nonrelativistic wave function turns out to be 4-dimensional. We derive the system for 4 functions. It is solved in terms of confluent hypergeometric functions. There arise three series of energy levels with corresponding solutions. This result agrees with that obtained for the relativistic Stückelberg equation.

Аннотация

Уравнение Штюкельберга для частицы с двумя спиновыми состояниями $S = 1$, $S = 0$ исследуется в присутствии внешнего однородного магнитного поля. В релятивистском случае частица описывается 11-компонентной волновой функцией. На решениях диагонализируются операторы энергии, третьей проекции полного углового момента и третьей проекции линейного момента вдоль направления магнитного поля. После разделения переменных получена система для 11 функций от одной переменной. В данной системе выполнено нерелятивистское приближение. При этом применяется известный метод получения нерелятивистских уравнений из релятивистских, основанный на проективных операторах, связанных с матрицей Γ_0 релятивистского уравнения. Нерелятивистская волновая функция оказывается четырехмерной. Получена система для четырех функций. Построены точные решения в вырожденных гипергеометрических функциях. Найдены три серии энергетических уровней, что согласуется с результатом, полученным для релятивистского уравнения Штюкельберга.

Keywords:

Stückelberg particle, nonrelativistic approximation, magnetic field, projective operators, exact solutions, bound states

Ключевые слова:

частица Штюкельберга, нерелятивистское приближение, магнитное поле, проективные операторы, точные решения, связанные состояния

Introduction

In previous paper [1], we studied the relativistic Stückelberg tensor system (see the references in [1]) of 11 equations in presence of the external uniform magnetic field. The relativistic particle is described by 11-component wave function, consisted of scalar, vector, and antisymmetric tensor. On solutions there are diagonalized operators of energy, the third projection of the total angular momentum, and the third projection of the linear momentum along the magnetic field direction. After separating the variables, the system of 11 radial functions was derived, and it was solved in the terms of confluent hypergeometric functions. Three series of the energy levels are found.

In the present paper we study the non-relativistic approximation for this problem. We apply the well-known method (see [2–4]) from the general theory of relativistic wave equations, based on the minimal equation for the matrix Γ_0 (in the model under consideration, it is an 11×11-matrix). This minimal equation allows us to introduce three projective operators P_+, P_-, P_0 and then expand the wave function into three components: $\Psi = \Psi_+ + \Psi_- + \Psi_0$. From the general theory it is known that when obtaining the nonrelativistic approximation the components Ψ_- and Ψ_0 should be considered as small, and Ψ_+ — as large ones. Only the component Ψ_+ enters the nonrelativistic equation. The nonrelativistic wave function turns out to be 4-dimensional. We derive the radial system for 4 functions. It is solved in terms of confluent hypergeometric functions. There arise three series of energy levels with corresponding solutions. This result agrees with that obtained for relativistic Stückelberg equation.

1. Nonrelativistic approximation and projective operators

We start with the matrix Γ_0 of the basic Stückelberg equation (see [1])

$$\Gamma^0 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (1)$$

It obeys the minimal equation $\Gamma(\Gamma^2+1)=0$, which permits us to define three projective operators

$$P_0 = 1+\Gamma_0^2, \quad P_1 = P_+ = \frac{1}{2} i\Gamma(i\Gamma+1),$$
$$P_2 = P_- = \frac{1}{2} i\Gamma(i\Gamma-1) \qquad (2)$$

with the properties

$$P_0^2 = P_0, \quad P_+^2 = P_+, \quad P_-^2 = P_-,$$
$$P_0 + P_+ + P_- = I.$$

In accordance with this, the complete wave function may be decomposed into the sum of three parts

$$\Psi = \Psi_+ + \Psi_- + \Psi_0,$$
$$\Psi_+ = P_+\Psi, \quad \Psi_- = P_-\Psi, \quad \Psi_0 = P_0\Psi. \qquad (3)$$

It is known from the general theory that in nonrelativistic approximation the component Ψ_+ should be considered as a big one, whereas the components Ψ_-, Ψ_0 — as small ones. We readily find their explicit structure:

$$\Psi_+ = \frac{1}{2}\Big(H - i\Psi_0, i(H-i\Psi_0), \Psi_1 - iE_1, \Psi_2 - iE_2,$$
$$\Psi_3 - iE_3, i(\Psi_1 - iE_1), i(\Psi_2 - iE_2), i(\Psi_3 - iE_3), 0, 0, 0\Big)^t =$$
$$= (L_0, iL_0, L_1, L_2, L_3, iL_1, iL_2, iL_3, 0, 0, 0)^t,$$
$$\Psi_- = \frac{1}{2}\Big(H + i\Psi_0, -i(H+i\Psi_0), \Psi_1 + iE_1, \Psi_2 + iE_2,$$
$$\Psi_3 + iE_3, -i(\Psi_1 + iE_1), -i(\Psi_2 + iE_2),$$
$$-i(\Psi_3 + iE_3), 0, 0, 0\Big)^t =$$
$$= \big(S_0, -iS_0, S_1, S_2, S_3, -iS_1, -iS_2, -iS_3, 0, 0, 0\big)^t,$$
$$\Psi_0 = (0,0,0,0,0,0,0,0,B_1,B_2,B_3)^t, \qquad (4)$$

where t stands for transpose. We have introduced special notations for big and small functions.

Because when solving the relativistic problem [1], we used the cyclic basis, now we also should transform big and small component to this basis. Because all blocks of the matrix Γ^0 preserve their form in cyclic basis,

$$\bar{\Delta}^0 = (1,0,0,0)^t, \quad -\bar{G}^0 = (-1,0,0,0),$$

$$\bar{K}^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{L}^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we conclude the rule for obtaining big and small components remains the same in this known basis:

$$\bar{P}_0 = 1 + \bar{\Gamma}_0^2,$$
$$\bar{P}_+ = \frac{1}{2} i\bar{\Gamma}(i\bar{\Gamma}+1), \quad \bar{P}_- = \frac{1}{2} i\bar{\Gamma}(i\bar{\Gamma}-1).$$

The formulas (4) may be written in more convenient variables as follows:

$$\Psi_+ = \frac{1}{2}\Big(h - ih_0, i(h-ih_0), h_1 - iE_1, h_2 - iE_2,$$
$$h_3 - iE_3, i(h_1 - iE_1), i(h_2 - iE_2), i(h_3 - iE_3), 0, 0, 0\Big)^t =$$
$$= (L_0, iL_0, L_1, L_2, L_3, iL_1, iL_2, iL_3, 0, 0, 0)^t,$$
$$\Psi_- = \frac{1}{2}\Big(h + ih_0, -i(h+ih_0), h_1 + iE_1, h_2 + iE_2,$$
$$h_3 + iE_3, -i(h_1 + iE_1), -i(h_2 + iE_2),$$
$$-i(h_3 + iE_3), 0, 0, 0\Big)^t =$$
$$= \big(S_0, -iS_0, S_1, S_2, S_3, -iS_1, -iS_2, -iS_3, 0, 0, 0\big)^t,$$

$$\Psi_0 = (0,0,0,0,0,0,0,0,B_1,B_2,B_3)^t. \qquad (5)$$

Hence we can derive inverse expressions for initial variables through big and small components:

$$h = \frac{1}{2}(L_0 + S_0), \quad h_0 = i\frac{1}{2}(L_0 - S_0),$$
$$h_i = \frac{1}{2}(L_i + S_i), \quad E_i = i\frac{1}{2}(L_i - S_i), \qquad (6)$$
$$i = 1, 2, 3.$$

Now we turn to relativistic system of equations (see in [1]), collecting them in 4 pairs and one triple:

$$-i\epsilon h_0 - ikh_2 + \frac{1}{\sqrt{2}}h_1' - \frac{(Br^2+2m-2)}{2\sqrt{2}r}h_1 - \frac{1}{\sqrt{2}}h_3' - \frac{(Br^2+2m+2)}{2\sqrt{2}r}h_3 = -\mu h,$$

$$-i\epsilon h - ikE_2 + \frac{1}{\sqrt{2}}E_1' - \frac{(Br^2+2m-2)}{2\sqrt{2}r}E_1 - \frac{1}{\sqrt{2}}E_3' - \frac{(Br^2+2m+2)}{2\sqrt{2}r}E_3 = \mu h_0;$$

$$-\frac{1}{\sqrt{2}}h' - \frac{m+Br^2/2}{\sqrt{2}r}h + \frac{1}{\sqrt{2}}B_2' + \frac{(Br^2+2m)}{2\sqrt{2}r}B_2 - ikB_3 + i\epsilon E_1 = \mu h_1,$$
$$+\frac{1}{\sqrt{2}}h_0' + \frac{(Br^2+2m)}{2\sqrt{2}r}h_0 - i\epsilon h_1 = \mu E_1;$$

$$ikh + i\epsilon E_2 - \frac{1}{\sqrt{2}}B_1' - \frac{(Br^2+2m+2)}{2\sqrt{2}r}B_1 - \frac{1}{\sqrt{2}}B_3' + \frac{(Br^2+2m-2)}{2\sqrt{2}r}B_3 = \mu h_2,$$
$$-ikh_0 - i\epsilon h_2 = \mu E_2;$$

$$\frac{1}{\sqrt{2}}h' - \frac{m+Br^2/2}{\sqrt{2}r}h + \frac{1}{\sqrt{2}}B_2' - \frac{(Br^2+2m)}{2\sqrt{2}r}B_2 + ikB_1 + i\epsilon E_3 = \mu h_3,$$
$$-\frac{1}{\sqrt{2}}h_0' + \frac{Br^2+2m}{2\sqrt{2}r}h_0 - i\epsilon h_3 = \mu E_3;$$
$$-\frac{1}{\sqrt{2}}h_2' + \frac{Br^2+2m}{2\sqrt{2}r}h_2 + ikh_3 = \mu B_1,$$
$$-ikh_1 - \frac{1}{\sqrt{2}}h_2' - \frac{Br^2+2m}{2\sqrt{2}r}h_2 = \mu B_3,$$

$$+\frac{1}{\sqrt{2}}h_1' - \frac{Br^2+2m-2}{2\sqrt{2}r}h_1 + \frac{1}{\sqrt{2}}h_3' + \frac{Br^2+2m+2}{2\sqrt{2}r}h_3 = \mu B_2.$$

Eliminating the small components B_1, B_2, B_3 with the help of three last equations we get

$$-i\epsilon h_0 - \frac{Br^2+2m-2}{2\sqrt{2}r}h_1 - ikh_2 - \frac{Br^2+2m+2}{2\sqrt{2}r}h_3 + \frac{1}{\sqrt{2}}\frac{dh_1}{dr} - \frac{1}{\sqrt{2}}\frac{dh_3}{dr} = -\mu h,$$

$$-i\epsilon h - ikE_2 + \frac{1}{\sqrt{2}}\frac{dE_1}{dr} - \frac{Br^2+2m-2}{2\sqrt{2}r}E_1 - \frac{1}{\sqrt{2}}\frac{dE_3}{dr} - \frac{Br^2+2m+2}{2\sqrt{2}r}E_3 = \mu h_0;$$

$$i\epsilon\mu E_1 - \frac{(Br^2+2m)\mu}{2\sqrt{2}r}h + \frac{-B^2r^4 + (-8k^2-4Bm)r^2 - 4(m-1)^2}{8r^2}h_1 + \frac{(Br^2+2m)ik}{2\sqrt{2}r}h_2 + \frac{B^2r^4 + 4B(m+1)r^2 + 4m^2 - 4}{8r^2}h_3 - \frac{\mu}{\sqrt{2}}\frac{dh}{dr} + \frac{1}{2r}\frac{dh_1}{dr} + \frac{ik}{\sqrt{2}}\frac{dh_2}{dr} + \frac{Br^2+2m+1}{2r}\frac{dh_3}{dr} + \frac{1}{2}\frac{d^2h_1}{dr^2} + \frac{1}{2}\frac{d^2h_3}{dr^2} = \mu^2 h_1,$$
$$\frac{(Br^2+2m)\mu}{2\sqrt{2}r}h_0 - i\epsilon\mu h_1 + \frac{\mu}{\sqrt{2}}\frac{dh_0}{dr} = \mu^2 E_1;$$

$$i\epsilon\mu E_2 + ik\mu h - \frac{(Br^2+2m-2)ik}{2\sqrt{2}r}h_1 - \frac{(Br^2+2m)^2}{4r^2}h_2 - \frac{(Br^2+2m+2)ik}{2\sqrt{2}r}h_3 + \frac{ik}{\sqrt{2}}\frac{dh_1}{dr} + \frac{1}{r}\frac{dh_2}{dr} - \frac{ik}{\sqrt{2}}\frac{dh_3}{dr} + \frac{d^2h_2}{dr^2} = \mu^2 h_2,$$
$$-ik\mu h_0 - i\epsilon\mu h_2 = \mu^2 E_2;$$

$$i\epsilon\mu E_3 - \frac{(Br^2+2m)\mu}{2\sqrt{2}r}h + \frac{B^2r^4 + 4B(m-1)r^2 + 4m^2 - 4}{8r^2}h_1 + \frac{(Br^2+2m)ik}{2\sqrt{2}r}h_2 + \frac{-B^2r^4 + (-8k^2-4Bm)r^2 - 4(m+1)^2}{8r^2}h_3 + \frac{\mu}{\sqrt{2}}\frac{dh}{dr} - \frac{Br^2+2m-1}{2r}\frac{dh_1}{dr} - \frac{ik}{\sqrt{2}}\frac{dh_2}{dr} + \frac{1}{2r}\frac{dh_3}{dr} + \frac{1}{2}\frac{d^2h_1}{dr^2} + \frac{1}{2}\frac{d^2h_3}{dr^2} = \mu^2 h_3,$$

$$\frac{(Br^2+2m)\mu}{2\sqrt{2}r}h_0 - i\epsilon\mu h_3 - \frac{\mu}{\sqrt{2}}\frac{dh_0}{dr} = \mu^2 E_3.$$

Let us take into account the formulas (2), this results in pair I

$$\epsilon(L_0-S_0) - \frac{Br^2+2m-2}{2\sqrt{2}r}(L_1+S_1) -$$

$$-ik(L_2+S_2) - \frac{Br^2+2m+2}{2\sqrt{2}r}(L_3+S_3) +$$

$$+\frac{1}{\sqrt{2}}\frac{d(L_1+S_1)}{dr} - \frac{1}{\sqrt{2}}\frac{d(L_3+S_3)}{dr} = -\mu(L_0+S_0),$$

$$-\epsilon(L_0+S_0) - ik(L_2-S_2) + \frac{1}{\sqrt{2}}\frac{d(L_1-S_1)}{dr} -$$

$$-\frac{Br^2+2m-2}{2\sqrt{2}r}(L_1-S_1) - \frac{1}{\sqrt{2}}\frac{d(L_3-S_3)}{dr} -$$

$$-\frac{Br^2+2m+2}{2\sqrt{2}r}(L_3-S_3) = \mu(L_0-S_0);$$

pair II

$$-\epsilon\mu(L_1-S_1) - \frac{(Br^2+2m)\mu}{2\sqrt{2}r}(L_0+S_0) +$$

$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m-1)^2}{8r^2}(L_1+S_1) +$$

$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}(L_2+S_2) +$$

$$+\frac{B^2r^4+4B(m+1)r^2+4m^2-4}{8r^2}(L_3+S_3) -$$

$$-\frac{\mu}{\sqrt{2}}\frac{d(L_0+S_0)}{dr} + \frac{1}{2r}\frac{d(L_1+S_1)}{dr} +$$

$$+\frac{ik}{\sqrt{2}}\frac{d(L_2+S_2)}{dr} + \frac{Br^2+2m+1}{2r}\frac{d(L_3+S_3)}{dr} +$$

$$+\frac{1}{2}\frac{d^2(L_1+S_1)}{dr^2} + \frac{1}{2}\frac{d^2(L_3+S_3)}{dr^2} = \mu^2(L_1+S_1),$$

$$\frac{(Br^2+2m)\mu}{2\sqrt{2}r}(L_0-S_0) - \epsilon\mu(L_1+S_1) +$$

$$+\frac{\mu}{\sqrt{2}}\frac{d(L_0-S_0)}{dr} = \mu^2(L_1-S_1);$$

pair III

$$-\epsilon\mu(L_2-S_2) + ik\mu(L_0+S_0) -$$

$$-\frac{(Br^2+2m-2)ik}{2\sqrt{2}r}(L_1+S_1) -$$

$$-\frac{(Br^2+2m)^2}{4r^2}(L_2+S_2) -$$

$$-\frac{(Br^2+2m+2)ik}{2\sqrt{2}r}(L_3+S_3) +$$

$$+\frac{ik}{\sqrt{2}}\frac{d(L_1+S_1)}{dr} + \frac{1}{r}\frac{d(L_2+S_2)}{dr} -$$

$$-\frac{ik}{\sqrt{2}}\frac{d(L_3+S_3)}{dr} + \frac{d^2(L_2+S_2)}{dr^2} = \mu^2(L_2+S_2),$$

$$-ik\mu(L_0-S_0) - \epsilon\mu(L_2+S_2) = \mu^2(L_2-S_2);$$

pair IV

$$-\epsilon\mu(L_3-S_3) - \frac{(Br^2+2m)\mu}{2\sqrt{2}r}(L_0+S_0) +$$

$$+\frac{B^2r^4+4B(m-1)r^2+4m^2-4}{8r^2}(L_1+S_1) +$$

$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}(L_2+S_2) +$$

$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m+1)^2}{8r^2}(L_3+S_3) +$$

$$+\frac{\mu}{\sqrt{2}}\frac{d(L_0+S_0)}{dr} - \frac{Br^2+2m-1}{2r}\frac{d(L_1+S_1)}{dr} -$$

$$-\frac{ik}{\sqrt{2}}\frac{d(L_2+S_2)}{dr} + \frac{1}{2r}\frac{d(L_3+S_3)}{dr} +$$

$$+\frac{1}{2}\frac{d^2(L_1+S_1)}{dr^2} + \frac{1}{2}\frac{d^2(L_3+S_3)}{dr^2} = \mu^2(L_3+S_3),$$

$$\frac{(Br^2+2m)\mu}{2\sqrt{2}r}(L_0-S_0) - \epsilon\mu(L_3+S_3) -$$

$$-\frac{\mu}{\sqrt{2}}\frac{d(L_0-S_0)}{dr} = \mu^2(L_3-S_3).$$

Within each pair, let us sum and subtract equations, this results in:

pair I

$$\frac{Br^2+2m-2}{\sqrt{2}r}L_1 + 2ikL_2 + \frac{Br^2+2m+2}{\sqrt{2}r}L_3 -$$

$$-\sqrt{2}\frac{dL_1}{dr} + \sqrt{2}\frac{dL_3}{dr} + 2\epsilon S_0 = 2\mu S_0,$$

$$2\epsilon L_0 + \frac{-Br^2-2m+2}{\sqrt{2}r}S_1 -$$

$$-2ikS_2 - \frac{Br^2+2m+2}{\sqrt{2}r}S_3 +$$

$$+\sqrt{2}\frac{dS_1}{dr} - \sqrt{2}\frac{dS_3}{dr} = -2\mu L_0;$$

pair II

$$\frac{(Br^2+2m)\mu}{2\sqrt{2}r}(L_0-S_0)-\epsilon\mu(L_1+S_1)+$$

$$+\frac{\mu}{\sqrt{2}}\left(\frac{dL_0}{dr}-\frac{dS_0}{dr}\right)-\epsilon\mu(L_1-S_1)-$$

$$-\frac{(Br^2+2m)\mu}{2\sqrt{2}r}(L_0+S_0)+$$

$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m-1)^2}{8r^2}(L_1+S_1)+$$

$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}(L_2+S_2)+$$

$$+\frac{B^2r^4+4B(m+1)r^2+4m^2-4}{8r^2}(L_3+S_3)-$$

$$-\frac{\mu}{\sqrt{2}}\left(\frac{dL_0}{dr}+\frac{dS_0}{dr}\right)+\frac{1}{2r}\left(\frac{dL_1}{dr}+\frac{dS_1}{dr}\right)+$$

$$+\frac{ik}{\sqrt{2}}\left(\frac{dL_2}{dr}+\frac{dS_2}{dr}\right)+\frac{Br^2+2m+1}{2r}\left(\frac{dL_3}{dr}+\frac{dS_3}{dr}\right)+$$

$$+\frac{1}{2}\left(\frac{d^2L_1}{dr^2}+\frac{d^2S_1}{dr^2}\right)+\frac{1}{2}\left(\frac{d^2L_3}{dr^2}+\frac{d^2S_3}{dr^2}\right)=2\mu^2L_1,$$

$$-\epsilon\mu(L_1-S_1)-\frac{(Br^2+2m)\mu}{2\sqrt{2}r}(L_0+S_0)+$$

$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m-1)^2}{8r^2}(L_1+S_1)+$$

$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}(L_2+S_2)+$$

$$+\frac{B^2r^4+4B(m+1)r^2+4m^2-4}{8r^2}(L_3+S_3)-$$

$$-\frac{\mu}{\sqrt{2}}\left(\frac{dL_0}{dr}+\frac{dS_0}{dr}\right)+\frac{1}{2r}\left(\frac{dL_1}{dr}+\frac{dS_1}{dr}\right)+$$

$$+\frac{ik}{\sqrt{2}}\left(\frac{dL_2}{dr}+\frac{dS_2}{dr}\right)+\frac{Br^2+2m+1}{2r}\left(\frac{dL_3}{dr}+\frac{dS_3}{dr}\right)+$$

$$+\frac{1}{2}\left(\frac{d^2L_1}{dr^2}+\frac{d^2S_1}{dr^2}\right)+\frac{1}{2}\left(\frac{d^2L_3}{dr^2}+\frac{d^2S_3}{dr^2}\right)-$$

$$-\frac{(Br^2+2m)\mu}{2\sqrt{2}r}(L_0-S_0)+\epsilon\mu(L_1+S_1)-$$

$$-\frac{\mu}{\sqrt{2}}\left(\frac{dL_0}{dr}-\frac{dS_0}{dr}\right)=2\mu^2S_1;$$

pair III

$$-\epsilon\mu(L_2-S_2)+ik\mu(L_0-S_0)-$$

$$-\frac{(Br^2+2m-2)ik}{2\sqrt{2}r}(L_1+S_1)-$$

$$-\frac{(Br^2+2m)^2}{4r^2}(L_2+S_2)-$$

$$-\frac{(Br^2+2m+2)ik}{2\sqrt{2}r}(L_3+S_3)+$$

$$+\frac{ik}{\sqrt{2}}\left(\frac{dL_1}{dr}+\frac{dS_1}{dr}\right)+\frac{1}{r}\left(\frac{dL_2}{dr}+\frac{dS_2}{dr}\right)-$$

$$-\frac{ik}{\sqrt{2}}\left(\frac{dL_3}{dr}+\frac{dS_3}{dr}\right)+\frac{d^2L_2}{dr^2}+\frac{d^2S_2}{dr^2}-$$

$$-ik\mu(L_0-S_0)-\epsilon\mu(L_2+S_2)=2\mu^2L_2;$$

$$-\epsilon\mu(L_2-S_2)+ik\mu(L_0+S_0)-$$

$$-\frac{(Br^2+2m-2)ik}{2\sqrt{2}r}(L_1+S_1)-$$

$$-\frac{(Br^2+2m)^2}{4r^2}(L_2+S_2)-$$

$$-\frac{(Br^2+2m+2)ik}{2\sqrt{2}r}(L_3+S_3)+$$

$$+\frac{ik}{\sqrt{2}}\left(\frac{dL_1}{dr}+\frac{dS_1}{dr}\right)+\frac{1}{r}\left(\frac{dL_2}{dr}+\frac{dS_2}{dr}\right)-$$

$$-\frac{ik}{\sqrt{2}}\left(\frac{dL_3}{dr}+\frac{dS_3}{dr}\right)+\frac{d^2L_2}{dr^2}+\frac{d^2S_2}{dr^2}+$$

$$+ik\mu(L_0-S_0)+\epsilon\mu(L_2+S_2)=2\mu^2S_2;$$

pair IV

$$-\epsilon\mu(L_3-S_3)-\frac{(Br^2+2m)\mu}{2\sqrt{2}r}(L_0+S_0)+$$

$$+\frac{B^2r^4+4B(m-1)r^2+4m^2-4}{8r^2}(L_1+S_1)+$$

$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}(L_2+S_2)+$$

$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m+1)^2}{8r^2}(L_3+S_3)+$$

$$+\frac{\mu}{\sqrt{2}}\left(\frac{dL_0}{dr}+\frac{dS_0}{dr}\right)-\frac{Br^2+2m-1}{2r}\left(\frac{dL_1}{dr}+\frac{dS_1}{dr}\right)-$$

$$-\frac{ik}{\sqrt{2}}\left(\frac{dL_2}{dr}+\frac{dS_2}{dr}\right)+\frac{1}{2r}\left(\frac{dL_3}{dr}+\frac{dS_3}{dr}\right)+$$

$$+\frac{1}{2}\left(\frac{d^2L_1}{dr^2}+\frac{d^2S_1}{dr^2}\right)+\frac{1}{2}\left(\frac{d^2L_3}{dr^2}+\frac{d^2S_3}{dr^2}\right)+$$

$$+\frac{(Br^2+2m)\mu}{2\sqrt{2}r}(L_0-S_0)-\epsilon\mu(L_3+S_3)-$$

$$-\frac{\mu}{\sqrt{2}}\left(\frac{dL_0}{dr}-\frac{dS_0}{dr}\right)=2\mu^2L_3,$$

$$-\epsilon\mu(L_3-S_3)-\frac{(Br^2+2m)\mu}{2\sqrt{2}r}(L_0+S_0)+$$
$$+\frac{B^2r^4+4B(m-1)r^2+4m^2-4}{8r^2}(L_1+S_1)+$$
$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}(L_2+S_2)+$$
$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m+1)^2}{8r^2}(L_3+S_3)+$$
$$+\frac{\mu}{\sqrt{2}}\left(\frac{dL_0}{dr}+\frac{dS_0}{dr}\right)-\frac{Br^2+2m-1}{2r}\left(\frac{dL_1}{dr}+\frac{dS_1}{dr}\right)-$$
$$-\frac{ik}{\sqrt{2}}\left(\frac{dL_2}{dr}+\frac{dS_2}{dr}\right)+\frac{1}{2r}\left(\frac{dL_3}{dr}+\frac{dS_3}{dr}\right)+$$
$$+\frac{1}{2}\left(\frac{d^2L_1}{dr^2}+\frac{d^2S_1}{dr^2}\right)+\frac{1}{2}\left(\frac{d^2L_3}{dr^2}+\frac{d^2S_3}{dr^2}\right)-$$
$$-\frac{(Br^2+2m)\mu}{2\sqrt{2}r}(L_0-S_0)+\epsilon\mu(L_3+S_3)+$$
$$+\frac{\mu}{\sqrt{2}}\left(\frac{dL_0}{dr}-\frac{dS_0}{dr}\right)=2\mu^2L_3.$$

The parameter μ relates to physical (positive) mass by the formula

$$\mu=-M.$$

Let us separate the rest energy M by formal change $\epsilon=M+E$, where E is nonrelativistic energy of the particle. The above equations become simpler:

pair I

$$\frac{Br^2+2m-2}{\sqrt{2}r}L_1+2ikL_2+\frac{Br^2+2m+2}{\sqrt{2}r}L_3-$$
$$-\sqrt{2}\frac{dL_1}{dr}+\sqrt{2}\frac{dL_3}{dr}+2(M+E)S_0=-2MS_0,$$

$$2(M+E)\underline{L_0}+\frac{-Br^2-2m+2}{\sqrt{2}r}S_1-2ikS_2-$$
$$-\frac{Br^2+2m+2}{\sqrt{2}r}S_3+\sqrt{2}\frac{dS_1}{dr}-\sqrt{2}\frac{dS_3}{dr}=\underline{2ML_0};$$

pair II

$$-\frac{(Br^2+2m)M}{2\sqrt{2}r}(L_0-S_0)+(M+E)M(L_1+S_1)-$$
$$-\frac{M}{\sqrt{2}}\left(\frac{dL_0}{dr}-\frac{dS_0}{dr}\right)+(M+E)M(L_1-S_1)+$$
$$+\frac{(Br^2+2m)M}{2\sqrt{2}r}(L_0+S_0)+$$
$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m-1)^2}{8r^2}(L_1+S_1)+$$
$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}(L_2+S_2)+$$
$$+\frac{B^2r^4+4B(m+1)r^2+4m^2-4}{8r^2}(L_3+S_3)+$$
$$+\frac{M}{\sqrt{2}}\left(\frac{dL_0}{dr}+\frac{dS_0}{dr}\right)+\frac{1}{2r}\left(\frac{dL_1}{dr}+\frac{dS_1}{dr}\right)+$$
$$+\frac{ik}{\sqrt{2}}\left(\frac{dL_2}{dr}+\frac{dS_2}{dr}\right)+$$
$$+\frac{Br^2+2m+1}{2r}\left(\frac{dL_3}{dr}+\frac{dS_3}{dr}\right)+$$
$$+\frac{1}{2}\left(\frac{d^2L_1}{dr^2}+\frac{d^2S_1}{dr^2}\right)+\frac{1}{2}\left(\frac{d^2L_3}{dr^2}+\frac{d^2S_3}{dr^2}\right)=2M^2L_1,$$

$$(M+E)M(L_1-S_1)+\frac{(Br^2+2m)M}{2\sqrt{2}r}(L_0+S_0)+$$
$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m-1)^2}{8r^2}(L_1+S_1)+$$
$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}(L_2+S_2)+$$
$$+\frac{B^2r^4+4B(m+1)r^2+4m^2-4}{8r^2}(L_3+S_3)+$$
$$+\frac{M}{\sqrt{2}}\left(\frac{dL_0}{dr}+\frac{dS_0}{dr}\right)+\frac{1}{2r}\left(\frac{dL_1}{dr}+\frac{dS_1}{dr}\right)+$$
$$+\frac{ik}{\sqrt{2}}\left(\frac{dL_2}{dr}+\frac{dS_2}{dr}\right)+\frac{Br^2+2m+1}{2r}\left(\frac{dL_3}{dr}+\frac{dS_3}{dr}\right)+$$
$$+\frac{1}{2}\left(\frac{d^2L_1}{dr^2}+\frac{d^2S_1}{dr^2}\right)+\frac{1}{2}\left(\frac{d^2L_3}{dr^2}+\frac{d^2S_3}{dr^2}\right)+$$
$$+\frac{(Br^2+2m)M}{2\sqrt{2}r}(L_0-S_0)-(M+E)M(L_1+S_1)+$$
$$+\frac{M}{\sqrt{2}}\left(\frac{dL_0}{dr}-\frac{dS_0}{dr}\right)=2M^2S_1;$$

pair III

$$(M+E)M(L_2-S_2)-ikM(L_0+S_0)-$$
$$-\frac{(Br^2+2m-2)ik}{2\sqrt{2}r}(L_1+S_1)-\frac{(Br^2+2m)^2}{4r^2}(L_2+S_2)-$$
$$-\frac{(Br^2+2m+2)ik}{2\sqrt{2}r}(L_3+S_3)+\frac{ik}{\sqrt{2}}\left(\frac{dL_1}{dr}+\frac{dS_1}{dr}\right)+$$
$$+\frac{1}{r}\left(\frac{dL_2}{dr}+\frac{dS_2}{dr}\right)-\frac{ik}{\sqrt{2}}\left(\frac{dL_3}{dr}+\frac{dS_3}{dr}\right)+$$
$$+\frac{d^2L_2}{dr^2}+\frac{d^2S_2}{dr^2}+ikM(L_0-S_0)+$$
$$+(M+E)M(L_2+S_2)=2M^2L_2,$$
$$(M+E)M(L_2-S_2)-ikM(L_0+S_0)-$$
$$-\frac{(Br^2+2m-2)ik}{2\sqrt{2}r}(L_1+S_1)-\frac{(Br^2+2m)^2}{4r^2}(L_2+S_2)-$$

$$-\frac{(Br^2+2m+2)ik}{2\sqrt{2}r}(L_3+S_3)+\frac{ik}{\sqrt{2}}\left(\frac{dL_1}{dr}+\frac{dS_1}{dr}\right)+$$
$$+\frac{1}{r}\left(\frac{dL_2}{dr}+\frac{dS_2}{dr}\right)-\frac{ik}{\sqrt{2}}\left(\frac{dL_3}{dr}+\frac{dS_3}{dr}\right)+$$
$$+\frac{d^2L_2}{dr^2}+\frac{d^2S_2}{dr^2}-ikM(L_0-S_0)-$$
$$-(M+E)M(L_2+S_2)=2M^2S_2;$$

pair IV

$$(M+E)M(L_3-S_3)+\frac{(Br^2+2m)M}{2\sqrt{2}r}(L_0+S_0)+$$
$$+\frac{B^2r^4+4B(m-1)r^2+4m^2-4}{8r^2}(L_1+S_1)+$$
$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}(L_2+S_2)+$$
$$\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m+1)^2}{8r^2}(L_3+S_3)-$$
$$-\frac{M}{\sqrt{2}}\left(\frac{dL_0}{dr}+\frac{dS_0}{dr}\right)-\frac{Br^2+2m-1}{2r}\left(\frac{dL_1}{dr}+\frac{dS_1}{dr}\right)-$$
$$-\frac{ik}{\sqrt{2}}\left(\frac{dL_2}{dr}+\frac{dS_2}{dr}\right)+\frac{1}{2r}\left(\frac{dL_3}{dr}+\frac{dS_3}{dr}\right)+$$
$$+\frac{1}{2r}\left(\frac{d^2L_1}{dr^2}+\frac{d^2S_1}{dr^2}\right)+\frac{1}{2r}\left(\frac{d^2L_3}{dr^2}+\frac{d^2S_3}{dr^2}\right)-$$
$$-\frac{(Br^2+2m)M}{2\sqrt{2}r}(L_0-S_0)+(M+E)M(L_3+S_3)+$$
$$+\frac{M}{\sqrt{2}}\left(\frac{dL_0}{dr}-\frac{dS_0}{dr}\right)=2M^2L_3,$$

$$(M+E)M(L_3-S_3)+\frac{(Br^2+2m)M}{2\sqrt{2}r}(L_0+S_0)+$$
$$+\frac{B^2r^4+4B(m-1)r^2+4m^2-4}{8r^2}(L_1+S_1)+$$
$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}(L_2+S_2)+$$
$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m+1)^2}{8r^2}(L_3+S_3)-$$
$$-\frac{M}{\sqrt{2}}\left(\frac{dL_0}{dr}+\frac{dS_0}{dr}\right)-\frac{Br^2+2m-1}{2r}\left(\frac{dL_1}{dr}+\frac{dS_1}{dr}\right)-$$
$$-\frac{ik}{\sqrt{2}}\left(\frac{dL_2}{dr}+\frac{dS_2}{dr}\right)+\frac{1}{2r}\left(\frac{dL_3}{dr}+\frac{dS_3}{dr}\right)+$$
$$+\frac{1}{2}\left(\frac{d^2L_1}{dr^2}+\frac{d^2S_1}{dr^2}\right)+\frac{1}{2}\left(\frac{d^2L_3}{dr^2}+\frac{d^2S_3}{dr^2}\right)+$$
$$+\frac{(Br^2+2m)M}{2\sqrt{2}r}(L_0-S_0)-(M+E)M(L_3+S_3)-$$
$$-\frac{M}{\sqrt{2}}\left(\frac{dL_0}{dr}-\frac{dS_0}{dr}\right)=2M^2S_3.$$

Let us neglect small components, so we obtain

$$\frac{Br^2+2m-2}{\sqrt{2}r}L_1+2ikL_2+\frac{Br^2+2m+2}{\sqrt{2}r}L_3-$$
$$-\sqrt{2}\left(\frac{dL_1}{dr}-\frac{dL_3}{dr}\right)+2S_0(M+E)=-2MS_0,$$
$$2EL_0+\frac{-Br^2-2m+2}{\sqrt{2}r}S_1-2ikS_2-$$
$$-\frac{Br^2+2m+2}{\sqrt{2}r}S_3+\sqrt{2}\left(\frac{dS_1}{dr}-\frac{dS_3}{dr}\right)=0,$$
$$\frac{(Br^2+2m)M}{\sqrt{2}r}S_0+\sqrt{2}M\frac{dS_0}{dr}+$$
$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m-1)^2}{8r^2}L_1+$$
$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}L_2+$$
$$+\frac{B^2r^4+4B(m+1)r^2+4m^2-4}{8r^2}L_3+$$
$$+\frac{1}{2r}\frac{dL_1}{dr}+\frac{ik}{\sqrt{2}}\frac{dL_2}{dr}+\frac{Br^2+2m+1}{2r}\frac{dL_3}{dr}+$$
$$+\frac{1}{2}\frac{d^2L_1}{dr^2}+\frac{1}{2}\frac{d^2L_3}{dr^2}+2MEL_1=0,$$
$$\frac{(Br^2+2m)M}{\sqrt{2}r}L_0+\sqrt{2}M\frac{dL_0}{dr}-2(M+E)MS_1+$$
$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m-1)^2}{8r^2}L_1+$$
$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}L_2+$$
$$+\frac{B^2r^4+4B(m+1)r^2+4m^2-4}{8r^2}L_3+$$
$$+\frac{1}{2r}\frac{dL_1}{dr}+\frac{ik}{\sqrt{2}}\frac{dL_2}{dr}+\frac{Br^2+2m+1}{2r}\frac{dL_3}{dr}+$$
$$+\frac{1}{2}\frac{d^2L_1}{dr^2}+\frac{1}{2}\frac{d^2L_3}{dr^2}=2M^2S_1,$$
$$2MEL_2-2ikMS_0-\frac{(Br^2+2m-2)ik}{2\sqrt{2}r}L_1-$$
$$-\frac{(Br^2+2m)^2}{4r^2}L_2-\frac{(Br^2+2m+2)ik}{2\sqrt{2}r}L_3+$$
$$+\frac{ik}{\sqrt{2}}\frac{dL_1}{dr}+\frac{1}{r}\frac{dL_2}{dr}-\frac{ik}{\sqrt{2}}\frac{dL_3}{dr}+\frac{d^2L_2}{dr^2}=0,$$
$$-2(M+E)MS_2-2ikML_0-\frac{(Br^2+2m-2)ik}{2\sqrt{2}r}L_1-$$
$$-\frac{(Br^2+2m)^2}{4r^2}L_2-\frac{(Br^2+2m+2)ik}{2\sqrt{2}r}L_3+$$

$$+\frac{ik}{\sqrt{2}}\frac{dL_1}{dr}+\frac{1}{r}\frac{dL_2}{dr}-\frac{ik}{\sqrt{2}}\frac{dL_3}{dr}+\frac{d^2L_2}{dr^2}=2M^2S_2,$$

$$\frac{(Br^2+2m)M}{\sqrt{2}r}S_0-\sqrt{2}M\frac{dS_0}{dr}+2ML_3E+$$
$$+\frac{B^2r^4+4B(m-1)r^2+4m^2-4}{8r^2}L_1+$$
$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}L_2+$$
$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m+1)^2}{8r^2}L_3-$$
$$-\frac{Br^2+2m-1}{2r}\frac{dL_1}{dr}-\frac{ik}{\sqrt{2}}\frac{dL_2}{dr}+$$
$$+\frac{1}{2r}\frac{dL_3}{dr}+\frac{1}{2}\frac{d^2L_1}{dr^2}+\frac{1}{2}\frac{d^2L_3}{dr^2}=0,$$

$$-\sqrt{2}M\frac{dL_0}{dr}+\frac{(Br^2+2m)M}{\sqrt{2}r}L_0-2(M+E)MS_3+$$
$$+\frac{B^2r^4+4B(m-1)r^2+4m^2-4}{8r^2}L_1+$$
$$+\frac{(Br^2+2m)ik}{2\sqrt{2}r}L_2+$$
$$+\frac{-B^2r^4+(-8k^2-4Bm)r^2-4(m+1)^2}{8r^2}L_3-$$
$$-\frac{Br^2+2m-1}{2r}\frac{dL_1}{dr}-\frac{ik}{\sqrt{2}}\frac{dL_2}{dr}+\frac{1}{2r}\frac{dL_3}{dr}+$$
$$+\frac{1}{2}\frac{d^2L_1}{dr^2}+\frac{1}{2}\frac{d^2S_1}{dr^2}+\frac{1}{2}\frac{d^2L_3}{dr^2}=2M^2S_3.$$

We assume that nonrelativistic energy may be neglected in comparison with rest energy $M+E\approx M$. Also from equations (1), (4), (6), (8) we express the small variables S_0, S_1, S_2, S_3 and substitute them in equations (2), (3), (5), (7). In this way, we obtain equations which contain only the big components

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+2ME-k^2-\frac{(m-1)^2}{r^2}-\right.$$
$$\left.-\frac{B^2r^2}{4}-Bm\right)L_1=0,\ \ L_1=N_1f_1;$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+2ME-k^2-\frac{m^2}{r^2}-\right.$$
$$\left.-\frac{B^2r^2}{4}-Bm\right)L_2=0,\ \ L_2=N_2f_3;$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+2ME-k^2-\frac{(m+1)^2}{r^2}-\right.$$
$$\left.-\frac{B^2r^2}{4}-Bm\right)L_3=0,\ \ L_3=N_3f_2;$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+2ME-k^2-\right.$$
$$\left.-\frac{m^2}{r^2}-\frac{B^2r^2}{4}-Bm\right)L_0+$$
$$+\frac{1}{2\sqrt{2}}\left(\frac{d}{dr}-\frac{m+Br^2/2-1}{r}\right)L_1+$$
$$+\frac{1}{2\sqrt{2}}\left(\frac{d}{dr}+\frac{m+Br^2/2+1}{r}\right)L_3=0.$$

The last equation may be re-written differently

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+2ME-k^2-\right.$$
$$\left.-\frac{m^2}{r^2}-\frac{B^2r^2}{4}-Bm\right)L_0+$$
$$+\frac{1}{2\sqrt{2}}(N_1b_{m-1}f_1+N_3a_{m+1}f_2)=0. \quad (7)$$

It is evident that $L_0=const\, f_3$, then eq. (7) takes on the form

$$N_1b_{m-1}f_1+N_3a_{m+1}f_2=0. \quad (8)$$

There exist differential constraints (see in [1])

$$b_{m-1}f_1=C_1f_3,\quad a_{m+1}f_2=C_2f_3,$$

they permit us to transform the previous relation to the following form (see [1])

$$N_1b_{m-1}f_1+N_3a_{m+1}f_2=$$
$$=(N_1C_1+N_3C_2)f_3=0\Rightarrow N_1C_1+N_3C_2=0.$$

Thus we have obtained 4 separate equations (only 3 equations are different, their solutions will be found in the next section)

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+2ME-k^2-\frac{(m-1)^2}{r^2}-\right.$$
$$\left.-\frac{B^2r^2}{4}-Bm\right]L_1=0,\quad L_1=N_1f_1;$$
$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+2ME-k^2-\frac{m^2}{r^2}-\right.$$
$$\left.-\frac{B^2r^2}{4}-Bm\right]L_2=0,\quad L_2=N_2f_3;$$
$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+2ME-k^2-\frac{(m+1)^2}{r^2}-\right.$$
$$\left.-\frac{B^2r^2}{4}-Bm\right]L_3=0,\quad L_3=N_3f_2;$$
$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+2ME-k^2-\frac{m^2}{r^2}-\right.$$
$$\left.-\frac{B^2r^2}{4}-Bm\right]L_0=0,\quad L_0=N_0f_3;$$

and the algebraic constraint

$$N_1C_1+N_3C_2=0.$$

Recall that (see [1])

$$C_1 = \sqrt{X - B}, \quad C_2 = \sqrt{X + B},$$
$$X = 2BN = 2ME - k^2. \tag{9}$$

General solution on the nonrelativistic equation consists of three components (the general multiplier $e^{im\phi}e^{ikz}$ is omitted):

$$\Psi = e^{-iE_1 t} N_1 \begin{pmatrix}1\\0\\0\\0\end{pmatrix} f_1 + e^{-iE_2 t} N_3 \begin{pmatrix}0\\0\\1\\0\end{pmatrix} f_2 +$$
$$+ e^{-iE_3 t} N_2 \begin{pmatrix}0\\1\\0\\0\end{pmatrix} f_3 + e^{-iE_3 t} N_0 \begin{pmatrix}0\\0\\0\\1\end{pmatrix} f_3(r).$$

We have 3 different series of energy levels, E_1, E_2, E_3 and 3 different wave functions. This result agrees with that obtained in relativistic case.

2. Solving the differential equations

The above three equations let us transform to the variable $x = Br^2/2$, $B > 0$:

$$\frac{d^2 L_1}{dx^2} + \frac{1}{x}\frac{dL_1}{dx} + \Big[-\frac{1}{4} + \frac{1}{2}\frac{2ME - k^2 - Bm}{Bx} - \\ -\frac{1}{4}\frac{(m-1)^2}{x^2}\Big] L_1 = 0, \tag{10}$$

$$\frac{d^2 L_2}{dx^2} + \frac{1}{x}\frac{dL_2}{dx} + \Big[-\frac{1}{4} + \frac{1}{2}\frac{2ME - k^2 - Bm}{Bx} - \\ -\frac{1}{4}\frac{m^2}{x^2}\Big] L_2 = 0, \tag{11}$$

$$\frac{d^2 L_3}{dx^2} + \frac{1}{x}\frac{dL_3}{dx} + \Big[-\frac{1}{4} + \frac{1}{2}\frac{2ME - k^2 - Bm}{Bx} - \\ -\frac{1}{4}\frac{(m+1)^2}{x^2}\Big] L_3 = 0. \tag{12}$$

Consider eq. (10):

$$L_1 = X^{a_1} e^{b_1 x} F_1,$$
$$x\frac{d^2 F_1}{dx^2} + (2a_1 + 1 + 2b_1 x)\frac{dF_1}{dx} +$$
$$+ \Big[\frac{1}{4}(4b_1^2 - 1)x + \frac{1}{4}\frac{4a_1^2 - (m-1)^2}{x} +$$
$$+ \frac{1}{4}\frac{8a_1 b_1 B + 4b_1 B + 4ME - 2k^2 - 2Bm}{B}\Big] F_1 = 0.$$

Imposing restrictions $a_1 = \pm\frac{1}{2}|m-1|$, $b_1 = -\frac{1}{2}$ we obtain the equation of confluent hypergeometric type

$$x\frac{d^2 F_1}{dx^2} + (2a_1 + 1 - x)\frac{dF_1}{dx} +$$
$$+ \frac{1}{2}\frac{-(2a_1+1)B + 2ME - k^2 - Bm}{B} F_1 = 0$$

with parameters

$$\alpha_1 = \frac{(2a_1+1)B - 2ME + k^2 + Bm}{2B}, \ \gamma_1 = 2a_1 + 1.$$

In order to get the bound states

$$a_1 = +\frac{1}{2}|m-1|, \ \gamma_1 = |m-1| + 1,$$

$$\alpha_1 = \frac{k^2 + B(|m-1| + m + 1) - 2ME}{2B};$$

that polynomial condition $\alpha_1 = -n_1$ gives

$$E_1 - \frac{k^2}{2M} = \frac{B}{M}\left(n_1 + \frac{|m-1| + m + 1}{2}\right). \tag{13}$$

Two other equations lead to similar results. Thus, we get

$$L_1 = x^{|m-1|/2} F^{-x/2} F(-n_1, |m-1| + 1, x),$$

$$E_1 - \frac{k^2}{2M} = \frac{B}{M}\left(n_1 + \frac{|m-1| + m + 1}{2}\right),$$

$$L_2 = x^{|m|/2} F^{-x/2} F(-n_2, |m| + 1, x),$$

$$E_2 - \frac{k^2}{2M} = \frac{B}{M}\left(n_2 + \frac{|m| + m + 1}{2}\right), \tag{14}$$

$$L_3 = x^{|m+1|/2} F^{-x/2} F(-n_3, |m+1| + 1, x),$$

$$E_3 - \frac{k^2}{2M} = \frac{B}{M}\left(n_3 + \frac{|m+1| + m + 1}{2}\right).$$

Discussion

The nonrelativistic wave function for Stückelberg particle turned out to be 4-dimensional. We have derived the corresponding radial system for 4 functions. It has been solved in terms of confluent hypergeometric functions. There arise three series of energy levels with corresponding solutions. This result agrees with that obtained for the relativistic Stückelberg equation.

It may be noted that the similar nonrelativistic study for Stückelberg particle in presence of the external Coulomb field was done in [5]. The energy spectrum was found. Besides, a general Pauli-like equation was derived for this particle in presence of arbitrary electromagnetic field.

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